The question of interior blow-up points for an elliptic Neumann problem : the critical case

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Abstract

In contrast with the subcritical case, we prove that for any bounded domain Ω in $\mathbb{R}^3$, the Neumann elliptic problem with critical nonlinearity

$$-\Delta u + \mu u = u^5\ ,\ u > 0 \text{ in } \Omega\ ;\ \partial u/\partial \nu = 0 \text{ on } \partial\Omega$$

has no solution blowing up at only interior points as μ goes to infinity.

1 Introduction and Results

Wondering about the mechanisms of pattern formation in biology, Turing [42] made the very important discovery that contrary to the intuition, which associates diffusion phenomena to a smoothing of initial data, spatial concentration structures may result from the interaction of two substances with different diffusion rates. Since that time many biological patterns have be explained in such a way. Models describing the evolution of the two involved substances concentrations, as those proposed by Keller and Segel, or Gierer and Meinhardt, consist in a system of two coupled nonlinear parabolic equations. Under some further assumptions, finding stationary solutions to the system reduces to solving a single nonlinear elliptic equation with Neumann boundary conditions [32]

$$(P_\mu) \begin{cases} -\Delta u + \mu u & = u^p\ ,\ u > 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} & = 0 \quad\quad\quad\quad\ \text{on } \partial\Omega \end{cases}$$

where $p > 1$, $\mu > 0$ are fixed parameters, and Ω is a smooth bounded domain in $\mathbb{R}^n$. Note that setting $v = \mu^{-1/(p-1)}u$, $d^2 = 1/\mu$, problem (P_μ) is equivalent to

$$(P'_d) \begin{cases} -d^2\Delta v + v & = v^p\ ,\ v > 0 \quad \text{in } \Omega \\ \frac{\partial v}{\partial \nu} & = 0 \quad\quad\quad\quad\ \text{on } \partial\Omega. \end{cases}$$

Since the works of Lin, Ni and Takagi [31, 32, 36, 37], many papers have been devoted to the study of (P'_d), under the assumption that p is subcritical, i.e. $n = 2$, or $n \geq 3$ and $p < (n+2)/(n-2)$. A natural question is to know whether the results which hold for subcritical exponents are true, or not, for critical or supercritical exponents. The study of the critical case made similarities and differences appear with respect to the subcritical case. For example, it was proved that when p is subcritical, the only solution to (P_μ) for small μ is constant [32], and the same holds when $n = 3$ and p is critical [52, 49]. However, if $n = 4, 5, 6$, Ω is a ball and p is critical, (P_μ) has at least one nonconstant radial solution for small μ [6].

For large μ, it is known that in both subcritical and critical cases, (P_μ) has solutions which concentrate at some points of the domain as μ goes to infinity (alternatively, d goes to zero in (P'_d)). The next question is to characterize such concentration points. In both cases, the least energy solutions have, for large μ, exactly one maximum point which lies on the boundary of the domain, and which goes, as μ goes to infinity, to a maximum point of the mean curvature of the boundary [36, 37] [4, 5, 35, 44, 40].

For subcritical exponents, higher energy solutions exist which blow up at one or several points of the boundary [21, 29, 49, 17] as μ goes to infinity. Solutions also exist which blow up at one or several points in the interior of the domain [48, 19, 16, 23, 27, 20, 51, 10]. In particular, (P_μ) has single interior spike solutions which blow up at a local maximum point of the distance function $d(x, \partial\Omega),\ x \in \Omega$. (Solutions have also been built which blow up at interior and boundary points at the same time [26].) For critical exponent, all the existence results concern solutions which blow up at one or several points of the boundary as μ goes to infinity [1-6, 12, 38, 40, 43-47, 33, 22, 24-25]. Hence the question : do solutions blowing up at interior points exist ?

The only known result is partial and negative [14] : for $n \geq 5$ and critical p, (P_μ) has no solution u_μ such that

$$\|u_\mu - \alpha_n \sum_{i=1}^{k} U_{\lambda^i_\mu, y^i_\mu}\|_{H^1(\Omega)} \to 0 \qquad \text{as } \mu \to \infty$$

with $\alpha_n = (n(n-2))^{(n-2)/4}$, $k \in \mathbb{N}^*$

$$U_{\lambda,y}(x) = \frac{\lambda^{(n-2)/2}}{(1+\lambda^2|x-y|^2)^{(n-2)/2}} \qquad \lambda \in \mathbb{R}^*_+,\ x, y \in \mathbb{R}^n \tag{1.1}$$

and $\lambda^i_\mu \to \infty$, $y^i_\mu \to y^i$ in Ω as $\mu \to \infty$, $y^i \neq y^j$ if $i \neq j$.

Such a result could have also been derived from the arguments in [38] and, in the case $n = 3$ and $k = 1$, from [40] (see the final remark at the end of Section 3). A recent paper shows that the case $k = 1$ cannot happen in any dimension [18].

Our aim in this paper is to consider the question of interior blow-up points, for $n = 3$ and critical p, without any assumption neither on the number of those points, nor on the distance between them, which may be zero.

$(u_\mu)_{\mu \geq \mu_0}$ being a sequence of nonconstant solutions to (P_μ), there are several and equivalent ways to define blow-up points of (u_μ). For example, $y \in \bar{\Omega}$ will be said to be a blow-up point of (u_μ) if and only if

$$\liminf_{r\to 0}\ \limsup_{\mu\to\infty}\int_{B(y,r)\cap\Omega}|\nabla u_\mu|^2\ \left(\text{or}\quad\int_{B(y,r)\cap\Omega}u_\mu^6\right)>0.$$

Our main result is :

Theorem 1 *Let $(u_\mu)_{\mu\geq\mu_0}$ a sequence, bounded in $H^1(\Omega)$, of solutions to (P_μ). There exists at least one blow-up point which lies on the boundary of Ω.*

We notice that, in contrast with the subcritical case, the existence of solutions with a finite number of blow-up points all lying in the interior of the domain, is excluded. We emphasize that the main difficulty in this work is to eliminate the possibility of *multiple* interior peaks without *a priori* assumption on the location of those peaks, which may be very close from each other, or even centered at the same point.

The next section is devoted to an *a priori* analysis of the solutions to (P_μ) as μ goes to infinity. This analysis, mixing together energy-dependent and energy-independent estimates, provides us with informations about the shape of solutions, which allow us to prove, through variational methods, the theorem in Section 3.

2 Blow-up analysis

2.1 Energy-independent estimates

We begin with an energy-independent description of the nonconstant solutions to (P_μ) as μ goes to infinity. We have the following proposition :

Proposition 2.1 *Let $(u_\mu)_{\mu\geq\mu_0}$ be a sequence of nonconstant solutions to (P_μ). Let $\varepsilon>0$, $R>1$. For μ large enough, u_μ has $N_\mu\in\mathbb{N}^*$ local maximum points $x_\mu^i\in\bar\Omega$, $1\leq i\leq N_\mu$, such that :*

$(i)\quad \dfrac{\mu^{1/4}}{u_\mu(x_\mu^i)}<\varepsilon$

$(ii)\quad \left\|\dfrac{1}{u_\mu(x_\mu^i)}u_\mu\!\left(\dfrac{x}{u_\mu^2(x_\mu^i)}+x_\mu^i\right)-\dfrac{1}{(1+\frac{|x|^2}{3})^{1/2}}\right\|_{C^2\left(B(0,2R)\cap\Omega_\mu^i\right)}<\varepsilon$

with $\Omega_\mu^i=u_\mu^2(x_\mu^i)(\Omega-x_\mu^i)$

$(iii)\quad x_\mu^i\in\partial\Omega\qquad$ *or* $\qquad u_\mu^2(x_\mu^i)d(x_\mu^i,\partial\Omega)>\dfrac{1}{\varepsilon}$

$(iv)\quad \bar B(x_\mu^i,r_\mu^i)\cap\bar B(x_\mu^j,r_\mu^j)=\emptyset\qquad$ *for* $i\neq j$, $\qquad r_\mu^i=\dfrac{R}{u_\mu^2(x_\mu^i)}$

$(v)\quad \Big(d\big(x,\{x_\mu^i,\ 1\leq i\leq N_\mu\}\big)\Big)^{1/2}u_\mu(x)\leq C(\varepsilon,R).$

(i) says that the maxima increase faster than $\mu^{1/4}$ as μ goes to infinity; (ii) describes the shape of u_μ in a neighbourhood of a maximum point x^i_μ; (iii) shows that either the maximum points x^i_μ are on the boundary, or are not too close from the boundary with respect to the heighth of the maximum; (iv) shows that these maximum points are not too close from each other with respect to their heighths, and (v) provides us with a global bound for u_μ in Ω.

Such a proposition relies on arguments initially developped by Schoen [41], in the context of the Yamabe problem. The proof, which follows the same scheme as in [30, Proposition 5.1], with the convenient additional arguments, is given in Appendix A.

Let now x_μ be any point in $\bar{\Omega}$. We set

$$v_\mu(y) = \frac{1}{\mu^{1/4}} u_\mu\Big(\frac{y}{\mu^{1/2}} + x_\mu\Big) \qquad y \in \Omega_\mu = \mu^{1/2}(\Omega - x_\mu) \tag{2.1}$$

which satisfies

$$-\Delta v_\mu + v_\mu = v_\mu^5 \ , \ v_\mu > 0 \text{ in } \Omega_\mu \ ; \quad \frac{\partial v_\mu}{\partial \nu} = 0 \text{ on } \partial\Omega_\mu. \tag{2.2}$$

Proposition 2.1 is equivalent to :

Proposition 2.2 *Let $\varepsilon > 0$, $R > 1$. For μ large enough, v_μ has $N_\mu \in \mathbb{N}^*$ local maximum points $y^i_\mu = \mu^{1/2}(x^i_\mu - x_\mu) \in \bar{\Omega}$, $1 \le i \le N_\mu$, such that :*

(i) $\quad v_\mu(y^i_\mu) > \dfrac{1}{\varepsilon}$

(ii) $$\left\| \frac{1}{v_\mu(y^i_\mu)} v_\mu\Big(\frac{y}{v^2_\mu(y^i_\mu)} + y^i_\mu\Big) - \frac{1}{(1+\frac{|y|^2}{3})^{1/2}} \right\|_{C^2\left(B(0,2R)\cap\Omega^i_\mu\right)} < \varepsilon$$

$$\textit{with } \Omega^i_\mu = v^2_\mu(y^i_\mu)(\Omega_\mu - y^i_\mu) = u^2_\mu(x^i_\mu)(\Omega - x^i_\mu)$$

(iii) $\quad y^i_\mu \in \partial\Omega \quad$ *or* $\quad v^2_\mu(y^i_\mu) d(y^i_\mu, \partial\Omega) > \dfrac{1}{\varepsilon}$

(iv) $\quad \bar{B}(y^i_\mu, s^i_\mu) \cap \bar{B}(y^j_\mu, s^j_\mu) = \emptyset \quad$ *for $i \neq j$,* $\quad s^i_\mu = \dfrac{R}{v^2_\mu(x^i_\mu)}$

(v) $$\Big(d\big(y, \{y^i_\mu,\ 1 \le i \le N_\mu\}\big)\Big)^{1/2} v_\mu(y) \le C(\varepsilon, R).$$

The interest of considering v_μ instead of u_μ is that v_μ solves a nonlinear elliptic equation (2.2) whose linear part has constant, hence bounded coefficients. This fact allows us to use techniques and results of Li, concerning the scalar curvature problem [28], Li and Zhu concerning Yamabe type equations [30].

In view of Proposition 2.2, we set :

Definition 1 *$\bar{y} \in \mathbb{R}^3$ is called an* isolated *blow-up point of (v_μ) if there exist $\bar{r} > 0$, $C \in \mathbb{R}$ and a sequence (y_μ) in $\bar{\Omega}_\mu$, converging to $\bar{y}$, such that y_μ is a local maximum of v_μ, $v_\mu(y_\mu) \to \infty$ and*

$$v_\mu(y) \le \frac{C}{|y - y_\mu|^{1/2}} \qquad y \in B(y_\mu, \bar{r}) \cap \Omega_\mu.$$

$\bar{y}$ is called a simple isolated *blow-up point if moreover there exists $r_0 > 0$ such that, for μ large enough, $r^{1/2}\bar{v}_\mu(r)$ has exactly one critical point in $(0, r_0)$, with*

$$\bar{v}_\mu(r) = \frac{1}{|\partial B(y_\mu, r) \cap \Omega_\mu|} \int_{\partial B(y_\mu, r) \cap \Omega_\mu} v_\mu \qquad 0 < r < \bar{r}.$$

Then, we can state :

Proposition 2.3

(i) Assume that $\bar{y}$ is an isolated blow-up point, such that for large μ

$$d(y_\mu, \partial\Omega_\mu) \ge \rho \tag{2.3}$$

for some $\rho > 0$. Then, $\bar{y}$ is a simple *isolated blow-up point.*

(ii) The same conclusion holds, without assuming (2.3), provided that Ω is convex.

Proposition 2.3(i) follows directly from [28, Section 3] or [30, Section 4], and (ii) follows from [52, Section 2.2]. Moreover, we know that under assumption (2.3), simple isolated blow-up points are such that

$$v_\mu(y_\mu) v_\mu(y) \le \frac{C}{|y - y_\mu|} \qquad y \in B(y_\mu, r_0) \cap \Omega_\mu \tag{2.4}$$

with C some positive constant independent of μ, and the same is true without assuming (2.3) if Ω is convex (see [28, Prop. 2.3][30, Prop. 3.1][52, Prop. 2.1]).

In view of our further needs, we consider the points y_μ^i defined in Proposition 2.2, and we prove the following proposition, which is of crucial interest in the sequel :

Proposition 2.4 *Let y_μ^i be as in Proposition 2.2.*

(i) Assume that there exists $\rho > 0$ such that, for large μ, $d(y_\mu^i, \partial\Omega_\mu) \ge \rho$, $1 \le i \le N_\mu$. Then, there exists $\delta > 0$ such that, for μ large enough

$$|y_\mu^i - y_\mu^j| \ge \delta \qquad \forall i, j \quad i \ne j. \tag{2.5}$$

(ii) If Ω is convex, (2.5) holds with $\rho = 0$.

This proposition still follows from [28, 30, 52]. Let us sketch the argument for (i). Assuming that the proposition is false, we may suppose, without loss of generality

$$\delta_\mu = |y_\mu^1 - y_\mu^2| = \min_{i \neq j} |y_\mu^i - y_\mu^j| \to 0 \text{ as } \mu \to \infty. \tag{2.6}$$

We set

$$w_\mu(z) = \delta_\mu^{1/2} v_\mu(\delta_\mu z + y_\mu^1) \qquad z \in \tilde{\Omega}_\mu = (\Omega_\mu - y_\mu^1)/\delta_\mu.$$

w_μ satisfies

$$-\Delta w_\mu + \delta_\mu^2 w_\mu = w_\mu^5 \ , \ w_\mu > 0 \text{ in } \tilde{\Omega}_\mu \ ; \quad \frac{\partial w_\mu}{\partial \nu} = 0 \text{ on } \partial\tilde{\Omega}_\mu \tag{2.7}$$

and, denoting $z_\mu^i = (y_\mu^i - y_\mu^1)/\delta_\mu$, $1 \leq i \leq N_\mu$, we know that

$$w_\mu(0), w_\mu(z_\mu^2) > R^{1/2} \quad \text{ because of Prop. 2.2}(iv) \tag{2.8}$$

$$|z_\mu^i - z_\mu^j| \geq 1 \qquad i \neq j \quad \text{ because of (2.6)} \tag{2.9}$$

$$\Big(d\big(y, \{z_\mu^i, \ 1 \leq i \leq N_\mu\}\big)\Big)^{1/2} w_\mu(y) \leq C \quad \text{ because of Prop. 2.2}(v). \tag{2.10}$$

Up to a subsequence, we may assume that $z_\mu^2 \to z^2$ as $\mu \to \infty$, $|z^2| = 1$. We claim that $w_\mu(0)$ and $w_\mu(z_\mu^2)$ go to infinity as μ goes to infinity.

Indeed, let us assume that $w_\mu(z_\mu^2)$ stays bounded. Prop. 2.2(ii) then implies that w_μ stays bounded in a fixed neighbourhood of z^2. Taking also account of (2.9-10), we see that w_μ stays bounded in any ball $B(z^2, r)$, $0 < r < 1$. If $w_\mu(0)$ goes to infinity, 0 is an isolated blow-up point, hence a simple isolated blow-up point, and an inequality as (2.4) holds for w_μ, i.e.

$$w_\mu(0) w_\mu(z) \leq \frac{C}{|z|} \qquad z \in B(0, r_0') \cap \tilde{\Omega}_\mu \tag{2.11}$$

Consequently, at a small and fixed distance of 0, w_μ goes to zero. Therefore, Harnack inequality applied to (2.7) shows that w_μ goes uniformly to zero in $B(z^2, r)$, $0 < r < 1$, in contradiction with (2.8).

If $w_\mu(0)$ stays bounded, either there is some z_μ^i, $i > 2$, such that $|z_\mu^i|$ stays bounded and $w_\mu(z_\mu^i)$ goes to infinity, and we can repeat the previous argument with z_μ^i instead of 0, whence again a contradiction; or w_μ stays bounded in any ball centered in 0. Then, elliptic theory shows that, up to a subsequence, w_μ goes in $C^2_{loc}(\mathbb{R}^3)$ to a limit w which satisfies $-\Delta w = w^5$, $w \geq 0$ in $\mathbb{R}^3$, $w \not\equiv 0$, $\nabla w(0) = \nabla w(z^2) = 0$. According to [13], such a w does not exist, hence again a contradiction. Therefore, 0 and z^2 are two simple isolated blow-up points.

Up to a reindexation and passing to a subsequence, we may assume that for $i \geq 2$, either $z_\mu^i \to z^i$, or $|z_\mu^i| \to \infty$ as $\mu \to \infty$. Because of (2.9), $|z^i - z^j| \geq 1$ if $i \neq j$. If $\bar{z}$ is a blow-up point for w_μ, $\bar{z} = z^i$ for some index i, because of (2.10). Let S be the set of these blow-up points, which are isolated and simple (note that $d(z_\mu^i, \partial\tilde{\Omega}_\mu) \geq \rho/\delta_\mu \to \infty$, and the equivalent of Prop. 2.3(i) holds for the solutions of (2.7), as [52] shows). We consider

$$\xi_\mu(z) = w_\mu(0) w_\mu(z)$$

which satisfies

$$-\Delta\xi_\mu + \delta_\mu^2 \xi_\mu = w_\mu^4 \xi_\mu \ , \ \xi_\mu > 0 \text{ in } \tilde{\Omega}_\mu \ ; \quad \frac{\partial \xi_\mu}{\partial \nu} = 0 \text{ on } \partial\tilde{\Omega}_\mu. \tag{2.12}$$

From (2.10-11) and Harnack inequality applied to (2.7), we know that w_μ goes uniformly to zero in any compact set $K \subset \mathbb{R}^3 \setminus S$ (note that $K \subset \tilde{\Omega}_\mu$ for μ large enough, since $d(0, \partial\tilde{\Omega}_\mu) \geq \rho/\delta_\mu$). From (2.11) and Harnack inequality applied to (2.12), we know that ξ_μ stays uniformly bounded in any compact set $K \subset \mathbb{R}^3 \setminus S$. Then, elliptic theory ensures that, along some subsequence, ξ_μ converges in $C^2_{loc}(\mathbb{R}^3 \setminus S)$ to a limit ξ, which is a positive regular harmonic function in $\mathbb{R}^3 \setminus S$. Therefore, we can write

$$\xi(z) = \frac{a}{|z|} + \frac{b}{|z - z^2|} + h(z)$$

with $a \geq 0$, $b \geq 0$ and h is regular positive harmonic function in $\mathbb{R}^3 \setminus (S - \{0, z^2\})$. 0 being a simple isolated blow-up point of (w_μ), $r \mapsto r^{1/2}\bar{\xi}_\mu(r)$ has a unique critical point in $(0, r_0')$, and Prop. 2.2(ii) shows that this function has a maximum point which goes to zero as μ goes to infinity. Therefore, $r \mapsto r^{1/2}\bar{\xi}(r)$ is nonincreasing in $(0, r_0')$. Then, either $\xi \equiv 0$, or $a > 0$. Integrating (2.12) on $B_{r_0'}$, we find

$$\int_{\partial B_{r_0'}} \frac{\partial \xi_\mu}{\partial \nu} + \delta_\mu^2 \int_{B_{r_0'}} \xi_\mu = w_\mu(0) \int_{B_{r_0'}} \xi_\mu^5.$$

From Prop. 2.2(ii), we have

$$\begin{aligned} w_\mu(0) \int_{B_{r_0'}} \xi_\mu^5 &\geq \frac{w_\mu(0)}{2} \int_{B(0, 2R/w_\mu^2(0))} \frac{w_\mu^5(0)}{(1 + \frac{w_\mu^4(0)|y|^2}{3})^{5/2}} dy \\ &\geq \frac{1}{2} \int_{B(0,2R)} \frac{dy}{(1 + \frac{|y|^2}{3})^{5/2}} \\ &\geq \tau \end{aligned}$$

with τ a strictly positive constant. On the other hand, (2.11) implies that

$$\delta_\mu^2 \int_{B_{r_0'}} \xi_\mu \leq C\delta_\mu^2 \int_{B_{r_0'}} \frac{dy}{|y|} = o(1)$$

as μ goes to infinity. Moreover, if $\xi \equiv 0$, we have also

$$\int_{\partial B_{r_0'}} \frac{\partial \xi_\mu}{\partial \nu} = o(1)$$

hence a contradiction. Consequently, $a > 0$. In the same way $b > 0$. The classical Pohozaev identity for (2.12) provides us with the equality

$$-\delta_\mu^2 \int_{B_\sigma(0)} \xi_\mu^2 = \int_{\partial B_\sigma(0)} \left(\frac{1}{2}\xi_\mu \frac{\partial \xi_\mu}{\partial \nu} - \frac{\sigma}{2}|\nabla \xi_\mu|^2 + \sigma(\frac{\partial \xi_\mu}{\partial \nu})^2 \right)$$
$$- \frac{1}{2} \int_{\partial B_\sigma(0)} \left(\frac{\delta_\mu^2}{2}\xi_\mu^2 - \frac{1}{6w_\mu^4(0)}\xi_\mu^6 \right)$$

for any small $\sigma > 0$. As δ_μ goes to zero and $w_\mu(0)$ goes to infinity, (2.11) implies that the left hand side, and the last integral on the right hand side, go to zero as μ goes to infinity. In the same time, a straightforward computation shows that

$$\lim_{\sigma \to 0} \lim_{\mu \to \infty} \int_{\partial B_\sigma(0)} \left(\frac{1}{2}\xi_\mu \frac{\partial \xi_\mu}{\partial \nu} - \frac{\sigma}{2}|\nabla \xi_\mu|^2 + \sigma(\frac{\partial \xi_\mu}{\partial \nu})^2 \right) \to -2\pi a(b + h(0)) < 0$$

whence a contradiction.

(ii), which is not necessary for our further purposes, may be proved in the same way, using the analysis of boundary blow-up points performed in [52].

Remark. If (u_μ) is assumed to be bounded in $H^1(\Omega)$, N_μ is also bounded, since

$$\|u_\mu\|_{H^1(\Omega)} \geq N_\mu \tau$$

where $\tau > 0$ is some fixed constant, as Prop. 2.1(ii) shows. Then, up to a reindexation and passing to a subsequence, we may assume that for μ large enough

$$N_\mu = k_1 + k_2 = k \in \mathbb{N}^*$$
$$d(y_\mu^i, \partial\Omega_\mu) \geq \rho \qquad \text{for some } \rho > 0 \qquad 1 \leq i \leq k_1$$
$$d(y_\mu^i, \partial\Omega_\mu) \to 0 \qquad \text{as } \mu \to \infty \qquad k_1 + 1 \leq i \leq k$$

Setting $\delta_\mu = \min_{\substack{1 \leq i,j \leq k_1 \\ i \neq j}} |y_\mu^i - y_\mu^j|$, the same arguments as previously show the existence of $\delta > 0$ such that $|y_\mu^i - y_\mu^j| > \delta$, $i \neq j$, $1 \leq i, j \leq k_1$.

2.2 Energy-dependent estimates

We turn now to an energy-dependent blow-up analysis of (u_μ), whose comparison with the previous results will provide us with the informations that we need to prove the theorem in the next section. First, as in [40], we define for $\lambda \in \mathbb{R}_+^*$ and $a \in \mathbb{R}^3$ the function

$$V_{\mu,\lambda,a}(x) = U_{\lambda,a}(x) - \varphi_{\mu,\lambda,a}(x) \qquad x \in \mathbb{R}^3 \tag{2.13}$$

where $U_{\lambda,a}$ is given by (1.3), i.e. $U_{\lambda,a} = \lambda^{1/2}(1+\lambda^2|x-a|^2)^{-1/2}$, and

$$\varphi_{\mu,\lambda,a}(x) = \frac{1-e^{-\mu^{1/2}|x-a|}}{\lambda^{1/2}|x-a|}. \tag{2.14}$$

$V_{\mu,\lambda,a}$, which satisfies in $\mathbb{R}^3$

$$-\Delta(3^{1/4}V_{\mu,\lambda,a}) + \mu(3^{1/4}V_{\mu,\lambda,a}) = (3^{1/4}V_{\mu,\lambda,a})^5 + \mu 3^{1/4}(U_{\lambda,a} - \frac{1}{\lambda^{1/2}|x-a|}) \tag{2.15}$$

is an improved approximate solution to (P_μ) with respect to $3^{1/4}U_{\lambda,a}$, as $\mu^{1/2}/\lambda$ goes to zero - see [40]. For $\mu^{1/2}/\lambda$ small, $\varphi_{\mu,\lambda,a}$ acts as a perturbation of $U_{\lambda,a}$ in $H^1(\Omega)$, since

$$\|\varphi_{\mu,\lambda,a}\|^2_{H^1(\Omega)} = O(\mu^{1/2}/\lambda) \tag{2.16}$$

as integral estimates show [40]. Now, we can state :

Proposition 2.5 *Let $(u_\mu)_{\mu\geq\mu_0}$ be a sequence of nonconstant solutions to (P_μ), bounded in $H^1(\Omega)$. There exist $k \in \mathbb{N}^*$, (λ^i_μ) and (a^i_μ) sequences in $\mathbb{R}^*_+$ and $\bar{\Omega}$ respectively such that, for some subsequence*

$$\|u_\mu - 3^{1/4}\sum_{i=1}^k V_{\mu,\lambda^i_\mu,a^i_\mu}\|_{H^1(\Omega)} \to 0 \qquad \text{as } \mu\to\infty \tag{2.17}$$

with

$$\mu^{1/2}/\lambda^i_\mu \to 0 \tag{2.18}$$

$$\lambda^i_\mu d(a^i_\mu, \partial\Omega) \to 0 \text{ or } \infty \tag{2.19}$$

$$\lambda^i_\mu/\lambda^j_\mu + \lambda^j_\mu/\lambda^i_\mu + \lambda^i_\mu\lambda^j_\mu|a^i_\mu - a^j_\mu|^2 \to \infty \qquad \text{if } i\neq j \tag{2.20}$$

as $\mu\to\infty$.

Note that Proposition 2.5 holds for Palais-Smale sequences as well, whereas the previous one applies to exact solutions of the equation only. Such an analysis is performed for the first time in [8], and [11]. A proof of it, following Bahri's arguments, is given in Appendix B.

In view of Proposition 2.5 we may assume, extracting some subsequence, that each of the sequences (a^i_μ) converges to a limit $a^i \in \bar{\Omega}$. It is easily seen that the a^i's are exactly the blow-up points of this subsequence.

2.3 The shape of solutions with only interior blow-up points

Let us assume that all the a^i's which occur in Proposition 2.5 lie in the interior of Ω. In order to prove the theorem, we have to prove that such a case cannot occur.

Comparing Proposition 2.5 and Proposition 2.2, Prop. 2.2(ii) implies that to each x_μ^i corresponds some $a_\mu^{j(i)}$ such that, ε being small

$$\lambda_\mu^{j(i)}/u_\mu^2(x_\mu^i) \quad \text{is close to } 1 \ ; \qquad \lambda_\mu^{j(i)}|a_\mu^{j(i)} - x_\mu^i| \quad \text{is close to } 0 \tag{2.21}$$

with $j(i_1) \neq j(i_2)$ if $i_1 \neq i_2$.

Conversely, Prop. 2.2(v) implies that to each a_μ^j corresponds some $x_\mu^{i(j)}$ such that $\lambda_\mu^j|a_\mu^j - x_\mu^{i(j)}|$ is bounded. We claim that for μ large enough, $i(j_1) \neq i(j_2)$ if $j_1 \neq j_2$.

Otherwise, up to subsequences and reindexations, we may assume that $j = 1, \ldots, p$ are the p indices, $p \geq 2$, such that $i(j) = 1$. We consider v_μ defined by (2.1), with $x_\mu = x_\mu^1$. Through the change of variable

$$x = \frac{y}{\mu^{1/2}} + x_\mu^1$$

$x = x_\mu^1$ is sent to $y = 0$, and $x = a_\mu^i$ is sent to $y = \mu^{1/2}(a_\mu^i - x_\mu^1)$. We know that 0 is an isolated blow-up point of (v_μ). Since x_μ^1 goes to $a^1 = \ldots = a^p$ which lies in the interior of Ω, $d(0, \Omega_\mu) \to \infty$, and Proposition 2.3 implies that 0 is a simple isolated blow-up point.

On the other hand, we notice that the λ_μ^j's are not of the same order as μ goes to infinity, that is

$$\lambda_\mu^i/\lambda_\mu^j + \lambda_\mu^j/\lambda_\mu^j \to \infty \qquad 0 \leq i < j \leq p.$$

Otherwise, $\lambda_\mu^j|a_\mu^j - x_\mu^1|$ being bounded, $\lambda_\mu^i\lambda_\mu^j|a_\mu^i - a_\mu^j|^2$ would also be bounded, and (2.20) could not be satisfied.

Finally, we notice that the boundedness of $\lambda_\mu^j|a_\mu^j - x_\mu^1|$ implies, through (2.18), that $\mu^{1/2}(a_\mu^j - x_\mu^1)$ goes to 0 as μ goes to infinity. It follows that $r \to r^{1/2}\bar{v}_\mu$, $r = |y - x_\mu^1|$, has several maximum points in any fixed interval $(0, r_0)$, for μ large enough. This contradicts the fact that 0 is a simple blow-up point.

Once we know that there is a correspondance one to one between the x_μ^i's and the a_μ^j's, we infer from (2.21) and Proposition 2.4 that there exists $\gamma > 0$ such that

$$|a_\mu^i - a_\mu^j| > \frac{\gamma}{\mu^{1/2}} \qquad i \neq j$$

for μ large enough. As a consequence, we know that a sequence (u_μ), bounded in $H^1(\Omega)$, of solutions to (P_μ) whose all blow-up points lie in the interior of Ω writes as

$$u_\mu = 3^{1/4} \sum_{i=1}^{k} V_{\mu, \lambda_\mu^i, a_\mu^i} + v_\mu \qquad k \in \mathbb{N}^* \tag{2.22}$$

with

$$\frac{\mu^{1/2}}{\lambda_\mu^i} \to 0 \ ; \qquad a_\mu^i \to a^i \in \Omega \qquad |a_\mu^i - a_\mu^j| > \frac{\gamma}{\mu^{1/2}} \quad \text{if } i \neq j \tag{2.23}$$

and
$$v_\mu \to 0 \quad \text{in } H^1(\Omega) \tag{2.24}$$
as μ goes to infinity.

We are going to prove, in the next section, that such a u_μ cannot solves (P_μ) for large μ - hence the theorem.

3 Proof of the theorem

We adopt in this section a variational approach of the problem. We define the functional
$$J_\mu(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \mu u^2) - \frac{1}{6}\int_\Omega u^6 \qquad u \in H^1(\Omega) \tag{3.1}$$
whose strictly positive critical points are exactly the solutions to (P_μ).

3.1 A parametrization of the variational problem

This subsection is devoted to a parametrization of the variational problem in a neighbourhood of the eventual solutions to (P_μ) defined by (2.22-24). $k \in \mathbb{N}^*$ and $\rho > 0$ being fixed, for $\varepsilon > 0$ we set

$$\mathcal{V}_{\varepsilon,\mu} = \Big\{ u \in H^1(\Omega) / \exists (\lambda_i)_{1\le i\le k} \in (\mathbb{R}_+^*)^k, \mu^{1/2}/\lambda_i < \varepsilon, \exists (a_i)_{1\le i\le k} \in (\Omega_\rho)^k,$$
$$|a_i - a_j| > \frac{\gamma}{4\mu^{1/2}} \text{ if } i \ne j, s.t.\ |\nabla(u - 3^{1/4}\sum_{i=1}^k V_{\mu,\lambda_i,a_i})|_2 < \varepsilon \Big\}$$

with $\Omega_\rho = \big\{x \in \Omega, d(x,\partial\Omega) > \rho\big\}$. Defining also

$$B_{\varepsilon,\mu} = \Big\{ (\alpha,\lambda,a) \in \mathbb{R}^k \times (R_+^*)^k \times (\Omega_{\rho-\varepsilon})^k \ s.t.\ 3^{1/4}/2 < \alpha_i < 2.3^{1/4},$$
$$\mu^{1/2}/\lambda_i < \varepsilon,\ |a_i - a_j| > \frac{\gamma - \varepsilon}{4\mu^{1/2}} \text{ if } i \ne j \Big\}$$

we have :

Lemma 3.1 *There exist $\mu_0 > 0$, $\varepsilon_0 > 0$ such that for any $\mu \ge \mu_0$, any ε, $0 < \varepsilon \le \varepsilon_0$, and any $u \in \mathcal{V}_{\varepsilon,\mu}$, the infimum*
$$\inf_{(\alpha,\lambda,y)\in B_{4\epsilon,\mu}} |\nabla(u - 3^{1/4}\sum_{i=1}^k V_{\mu,\lambda_i,a_i}|_2$$
is achieved at only one point, which lies in $B_{2\varepsilon,\mu}$.

Such a lemma is proved, for $k=1$, in [40, App. A]. The result extends easily to the case $k>1$, proceeding as in [8, Prop. 7][38, App. A]. Then, for $\lambda \in (\mathbb{R}_+^*)^k$ and $a \in \Omega^k$, we set

$$E_{\lambda,a,\mu} = \Big\{ v \in H^1(\Omega) / \int_\Omega \nabla v . \nabla V_\mu^i = \int_\Omega \nabla v . \nabla \tfrac{\partial V_\mu^i}{\partial \lambda_i} = \int_\Omega \nabla v . \nabla \tfrac{\partial V_\mu^i}{\partial (a_i)_l} = 0 \quad 1 \le i \le k, 1 \le l \le 3 \Big\} \tag{3.2}$$

with $V_\mu^i = V_{\mu,\lambda_i,a_i}$, for sake of simplicity. For $\mu \ge \mu_0$, Lemma 3.1 induces a map Φ from the open subset $\mathcal{V}_{\varepsilon_0,\mu}$ of $H^1(\Omega)$ to the manifold

$$\mathcal{M}_\mu = \Big\{ (\alpha,\lambda,a,v) \in \mathbb{R}^k \times (R_+^*)^k \times \Omega^k \times H^1(\Omega) \ s.t. \ (\alpha,\lambda,a) \in B_{2\varepsilon_0,\mu}, \quad v \in E_{\lambda,a,\mu}, |\nabla v|_2 < \varepsilon \Big\}$$

where $(\alpha(u),\lambda(u),a(u))$ is the unique point in $B_{2\varepsilon_0,\mu}$ at which the infimum of $|\nabla(u - 3^{1/4}\sum_{i=1}^k V_{\mu,\lambda_i,a_i}|_2$ is achieved, and $v(u) = u - \sum_{i=1}^k \alpha_i(u) V_{\mu,\lambda_i(u),a_i(u)}$. This map is open, and induces a diffeomorphism between $\mathcal{V}_{\varepsilon_0,\mu}$ and its image, which contains

$$\mathcal{N}_\mu = \Big\{ (\alpha,\lambda,a,v) \in \mathbb{R}^k \times (R_+^*)^k \times \Omega_\rho^k \times H^1(\Omega) \ s.t. |\alpha_i - 3^{1/4}| < \eta_0, \quad \mu^{1/2}/\lambda_i < \eta_0, \ |a_i - a_j| > \tfrac{\gamma}{2\mu^{1/2}} \text{ if } i \ne j, \ v \in E_{\lambda,a,\mu} \text{ and } |\nabla v|_2 < \varepsilon \Big\} \tag{3.3}$$

for some $\eta_0 > 0$ small enough. Setting

$$\begin{array}{rll} K_\mu : & \mathcal{N}_\mu & \to \mathbb{R} \\ & (\alpha,\lambda,a,v) & \mapsto J_\mu(\sum_{i=1}^k \alpha_i V_{\mu,\lambda_i,a_i} + v) \end{array} \tag{3.4}$$

we know that $(\alpha,\lambda,a,v) \in \mathcal{N}_\mu$ is a critical point of K_μ if and only if $u = \sum_{i=1}^k \alpha_i V_{\mu,\lambda_i,a_i} + v$ is a critical point of J_μ. Let us notice that, for μ large enough, u_μ given by (2.22-24) is in $\mathcal{V}_{\varepsilon_0,\mu}$. Moreover, setting

$$\Phi(u_\mu) = (\tilde{\alpha}_\mu, \tilde{\lambda}_\mu, \tilde{a}_\mu, \tilde{v}_\mu)$$

it follows from [8,38, Lemma A.1] that

$$\tilde{\alpha}_\mu^i \to 3^{1/4} \qquad \mu^{1/2}/\tilde{\lambda}_\mu \to 0 \qquad \tilde{a}_\mu^i \to a_i \qquad |\tilde{a}_\mu^i - \tilde{a}_\mu^j| > \frac{\gamma}{2\mu^{1/2}} \text{ if } i \ne j$$

and

$$\tilde{v}_\mu \to 0 \quad \text{in } H^1(\Omega).$$

In particular, $\Phi(u_\mu) \in \mathcal{N}_\mu$ for μ large enough. We are going to show that $(\tilde{\alpha}_\mu, \tilde{\lambda}_\mu, \tilde{a}_\mu, \tilde{v}_\mu)$ cannot be a critical point of K_μ for μ large enough, whence the theorem.

3.2 The v-derivative of K_μ

In this subsection, we estimate the H^1-norm of v_μ as μ goes to infinity. In view of (3.3-4), expanding K_μ with respect to v in a neighbourhood of $v = 0$, we find

$$K_\mu(\alpha, \lambda, a, v) = K_\mu(\alpha, \lambda, a, 0) + f_{\alpha,\lambda,a,\mu}(v) + Q_{\alpha,\lambda,a,\mu}(v) + R_{\alpha,\lambda,a,\mu}(v) \quad (3.5)$$

with

$$f_{\alpha,\lambda,a,\mu}(v) = \mu \int_\Omega (\sum_{i=1}^k \alpha_i V_\mu^i) v - \int_\Omega (\sum_{i=1}^k \alpha_i V_\mu^i)^5 v \quad (3.6)$$

$$Q_{\alpha,\lambda,a,\mu}(v) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + \mu u^2) - \frac{5}{2} \int_\Omega (\sum_{i=1}^k \alpha_i V_\mu^i)^4 v^2 \quad (3.7)$$

$$R_{\alpha,\lambda,a,\mu}(v) = O(\|v\|^3_{H^1(\Omega)}). \quad (3.8)$$

Moreover, choosing some $\eta_0 > 0$ sufficiently small, there exist $\kappa > 0$, $\kappa' > 0$ such that for μ large enough and any $(\alpha, \lambda, a, v) \in \mathcal{N}_\mu$

$$\kappa \int_\Omega (|\nabla u|^2 + \mu u^2) \le Q_{\alpha,\lambda,a,\mu}(v) \le \kappa' \int_\Omega (|\nabla u|^2 + \mu u^2). \quad (3.9)$$

The second inequality is a direct consequence of Hölder inequality, Sobolev embedding theorem and estimate (C.3) in appendix. The coercivity property follows from [40, Lemma 3.2] in the case $k = 1$. The result extends to the case $k > 1$ using the arguments of [7, Prop. 3.1], which are valid provided that $\lambda_i \lambda_j |a_i - a_j|^2$, $i \neq j$, is large enough. But $(\alpha, \lambda, a, v) \in \mathcal{N}_\mu$ implies that $\lambda_i \lambda_j |a_i - a_j|^2 > \gamma^2 / 4\eta_0^2$, whence the desired result choosing η_0 small enough.

On the other hand, we claim that there exists $C > 0$ such that, for μ large enough and any $(\alpha, \lambda, a, v) \in \mathcal{N}_\mu$, we have

$$|f_{\alpha,\lambda,a,\mu}(v)| \le \left(\frac{1}{\mu^{1/4} |\lambda|^{1/2}} + \frac{\mu^{1/2}}{|\lambda|} \right) \left(\frac{1}{2} \int_\Omega (|\nabla u|^2 + \mu u^2) \right)^{1/2} \quad (3.10)$$

with $|\lambda| = (\sum_{i=1}^k \lambda_i^2)^{1/2}$. Let us assume that the claim is true. Then, we deduce from (3.8-10) and the implicit functions theorem the following proposition :

Proposition 3.1 *There exist $\eta_1 > 0$, $\eta_2 > 0$ such that, for μ large enough, there exists a smooth map*

$$\begin{aligned} \tilde{\mathcal{N}_\mu} = \Big\{ &(\alpha, \lambda, a) \in \mathbb{R}^k \times (R_+^*)^k \times \Omega_\rho^k \ s.t. |\alpha_i - 3^{1/4}| < \eta_1, \\ &\mu^{1/2}/\lambda_i < \eta_1, \ |a_i - a_j| > \tfrac{\gamma}{2\mu^{1/2}} \ if \ i \neq j \Big\} \ \to E_{\lambda,a,\mu} \\ &(\alpha, \lambda, a) \ \mapsto \bar{v}_\mu(\alpha, \lambda, a) \end{aligned}$$

such that $\bar{v}_\mu(\alpha, \lambda, a)$ is the unique point $v \in E_{\lambda,a,\mu}$, $|\nabla v|_2^2 + \mu |v|_2^2 < \eta_2$, satisfying

$$\frac{\partial K_\mu}{\partial v} (\alpha, \lambda, a, \bar{v}_\mu(\alpha, \lambda, a)) = 0 \quad in \ T_{(\alpha,\lambda,a,\bar{v}_\mu)} \mathcal{N}_\mu. \quad (3.11)$$

Moreover, there exists $C > 0$ such that

$$\int_\Omega |\nabla \bar{v}_\mu|^2 + \mu \int_\Omega \bar{v}_\mu^2 \leq C \left(\frac{1}{\mu^{1/2}|\lambda|} + \frac{\mu}{|\lambda|}^2 \right). \tag{3.12}$$

We notice that (3.11) means that there exist $(A, B, C) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$ such that, for $w \in H^1(\Omega)$

$$\begin{aligned}
&\frac{\partial K_\mu}{\partial v}(\alpha, \lambda, a, \bar{v}_\mu).w \\
&\quad = \sum_{i=1}^k \left(A_i \int_\Omega \nabla V_\mu^i . \nabla w + B_i \int_\Omega \nabla \frac{\partial V_\mu^i}{\partial \lambda_i}.\nabla w + \sum_{l=1}^3 C_{il} \int_\Omega \nabla \frac{\partial V_\mu^i}{\partial (a_i)_l}.\nabla w \right).
\end{aligned}$$

Taking respectively $w = V_\mu^i,\ \frac{\partial V_\mu^i}{\partial \lambda_i},\ \frac{\partial V_\mu^i}{\partial (a_i)_l},\ 1 \leq i \leq k,\ 1 \leq l \leq 3$, we see that the $A_i,\ B_i,\ C_{il}$'s solve a linear system which is nearly diagonal since, using the integral estimates in [7, 40] and Appendix C, we have

$$\begin{aligned}
&\int_\Omega \nabla V_\mu^i . \nabla V_\mu^j = \frac{3\pi^2}{4}\delta_{ij} + O\Big(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}} \Big) \\
&\int_\Omega \nabla V_\mu^i . \nabla \frac{\partial V_\mu^j}{\partial \lambda_j} = O\Big(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{3/2}} \Big) \\
&\int_\Omega \nabla V_\mu^i . \nabla \frac{\partial V_\mu^j}{\partial (a_j)_l} = O\Big(\frac{\mu^{1/2}\lambda_j^{1/2}}{\lambda_i^{1/2}} \Big) \\
&\int_\Omega \nabla \frac{\partial V_\mu^i}{\partial \lambda_i} . \nabla \frac{\partial V_\mu^j}{\partial \lambda_j} = \frac{15\pi^2}{64}\frac{\delta_{ij}}{\lambda_i^2} + O\Big(\frac{\mu^{1/2}}{\lambda_i^{3/2}\lambda_j^{3/2}} \Big) \\
&\int_\Omega \nabla \frac{\partial V_\mu^i}{\partial \lambda_i} . \nabla \frac{\partial V_\mu^j}{\partial (a_j)_l} = O\Big(\frac{\mu^{1/2}\lambda_j^{1/2}}{\lambda_i^{3/2}} \Big) \\
&\int_\Omega \nabla \frac{\partial V_\mu^i}{\partial (a_i)_l} . \nabla \frac{\partial V_\mu^j}{\partial (a_j)_m} = \frac{15\pi^2}{64}\lambda_i^2\delta_{ij}\delta_{lm} + O\Big(\frac{\mu^{1/2}\lambda_j^{1/2}}{\lambda_i^{3/2}} \Big).
\end{aligned} \tag{3.13}$$

On the other hand

$$\begin{aligned}
&\frac{\partial K_\mu}{\partial v}(\alpha, \lambda, a, \bar{v}_\mu).V_\mu^i = \frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial \alpha_i}(\alpha, \lambda, a, \bar{v}_\mu) = O\Big(|\alpha_i - 3^{1/4}| + \frac{\mu^{1/2}}{|\lambda|} \Big) \\
&\frac{\partial K_\mu}{\partial v}(\alpha, \lambda, a, \bar{v}_\mu).\frac{V_\mu^i}{\partial \lambda_i} = \frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial \lambda_i}(\alpha, \lambda, a, \bar{v}_\mu) = O\Big(\frac{\mu^{1/2}}{\lambda_i|\lambda|} \Big) \\
&\frac{\partial K_\mu}{\partial v}(\alpha, \lambda, a, \bar{v}_\mu).\frac{V_\mu^i}{\partial (a_i)_l} = \frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial (a_i)_l}(\alpha, \lambda, a, \bar{v}_\mu) = O\Big(\frac{\mu^{1/2}\lambda_i}{|\lambda|} \Big)
\end{aligned}$$

as it follows from Proposition C.1 in appendix. Then, solving the linear system provides us with the estimates

$$\left.\begin{aligned} A_i &= O\Big(|\alpha_i - 3^{1/4}| + \frac{\mu^{1/2}}{|\lambda|}\Big) \\ B_i &= O\Big(\frac{\mu^{1/2}\lambda_i}{|\lambda|}\Big) \\ C_{il} &= O\Big(\frac{\mu^{1/2}}{\lambda_i|\lambda|}\Big) \end{aligned}\right\} \tag{3.14}$$

Before ending this subsection, we prove claim (3.10). From the proof of Lemma 3.1 in [40], we know that there exists $C > 0$ such that, for μ large enough and any $(\alpha, \lambda, a, v) \in \mathcal{N}_\mu$

$$\begin{aligned} &\left|\mu \int_\Omega \alpha_i V_\mu^i v - \int_\Omega (\alpha_i V_\mu^i)^5 v\right| \\ &\qquad \leq C \left(\frac{1}{\mu^{1/4}|\lambda|^{1/2}} + \frac{\mu^{1/2}}{|\lambda|}\right) \left(\frac{1}{2}\int_\Omega (|\nabla u|^2 + \mu u^2)\right)^{1/2} \end{aligned} \tag{3.15}$$

$1 \leq i \leq k$. On the other hand, if $i \neq j$, the Hölder inequality and the Sobolev embedding theorem yield

$$\int_\Omega (V_\mu^i)^4 V_\mu^j v \leq C\|v\|_{H^1(\Omega)} \left(\int_\Omega |V_\mu^i|^{24/5} |V_\mu^j|^{6/5}\right)^{5/6}. \tag{3.16}$$

(3.15-16) and estimate (C.16) in Appendix show that (3.10) is satisfied.

3.3 The α-derivative of K_μ

For $(\alpha, \lambda, a) \in \tilde{\mathcal{N}}_\mu$, we set

$$\tilde{K}_\mu(\alpha, \lambda, a) = K_\mu(\alpha, \lambda, a, \bar{v}_\mu(\alpha, \lambda, a)).$$

From the definition (3.1) of $E_{\lambda,a,\mu}$, it follows that the partial derivative of $\bar{v}_\mu$ with respect to α_i is also in $E_{\lambda,a,\mu}$. Therefore, we deduce from (3.11) that

$$\frac{\partial \tilde{K}_\mu}{\partial \alpha_i}(\alpha, \lambda, a) = \frac{\partial K_\mu}{\partial \alpha_i}(\alpha, \lambda, a, \bar{v}_\mu(\alpha, \lambda, a))$$

that is, according to Proposition C.1 in appendix

$$\frac{\partial \tilde{K}_\mu}{\partial \alpha_i}(\alpha, \lambda, a) = \frac{\pi^2}{4}\alpha_i(3 - \alpha_i^4) + O(\frac{\mu^{1/2}}{|\lambda|}).$$

In the same way, we have

$$\frac{\partial^2 \tilde{K}_\mu}{\partial \alpha_i^2}(\alpha, \lambda, a) = \frac{\partial^2 K_\mu}{\partial \alpha_i^2}(\alpha, \lambda, a, \bar{v}_\mu(\alpha, \lambda, a))$$

whence, according to (C.22)

$$\frac{\partial^2 \tilde{K}_\mu}{\partial \alpha_i^2}(\alpha,\lambda,a) = \frac{\pi^2}{4}(3-5\alpha_i^4) + O(\frac{\mu^{1/2}}{|\lambda|}).$$

Consequently, we obtain :

Proposition 3.2 *For μ large enough, there exists a smooth map*

$$\begin{aligned} \tilde{\tilde{\mathcal{N}}}_\mu = \Big\{ (\lambda,a) \in (R_+^*)^k \times \Omega_\rho^k \ s.t.\ \mu^{1/2}/\lambda_i < \eta_1 & \\ |a_i - a_j| > \tfrac{\gamma}{2\mu^{1/2}} \ if\ i \neq j \Big\} & \quad \to \mathbb{R}_+^k \\ (\lambda,a) & \quad \mapsto \bar{\alpha}_\mu(\lambda,a) \end{aligned}$$

such that $\bar{\alpha}_\mu(\lambda,a)$ is the unique point $\alpha \in \mathbb{R}_+^k$, $|\alpha_i - 3^{1/4}| < \eta_1$, satisfying

$$\frac{\partial \tilde{K}_\mu}{\partial \alpha}(\bar{\alpha}_\mu(\lambda,a),\lambda,a) = 0.$$

Moreover, $\bar{\alpha}_\mu(\lambda,a)$ satisfies

$$\bar{\alpha}_\mu^i(\lambda,a) = 3^{1/4} + O(\frac{\mu^{1/2}}{|\lambda|}). \tag{3.17}$$

3.4 The λ-derivative of K_μ

This last subsection will provide us with the contradiction which proves the theorem. For $(\lambda,a) \in \tilde{\tilde{\mathcal{N}}}_\mu$, we set

$$\tilde{\tilde{K}}_\mu(\lambda,a) = \tilde{K}_\mu\Big(\bar{\alpha}_\mu(\lambda,a),\lambda,a,\bar{v}_\mu(\bar{\alpha}_\mu(\lambda,a),\lambda,a)\Big).$$

From (3.11) we know that

$$\begin{aligned} &\frac{\partial \tilde{\tilde{K}}_\mu}{\partial \lambda_i}(\lambda,a) = \frac{\partial \tilde{K}_\mu}{\partial \lambda_i}(\bar{\alpha}_\mu,\lambda,a,\bar{v}_\mu) \\ &+\sum_{j=1}^{k}\left(A_j \int_\Omega \nabla V_\mu^j.\nabla \frac{\partial \bar{v}_\mu}{\partial \lambda_i} + B_j \int_\Omega \nabla \frac{\partial V_\mu^j}{\partial \lambda_j}.\nabla \frac{\partial \bar{v}_\mu}{\partial \lambda_i} + \sum_{l=1}^{3} C_{jl} \int_\Omega \nabla \frac{\partial V_\mu^j}{\partial (a_i)_l}.\nabla \frac{\partial \bar{v}_\mu}{\partial \lambda_i} \right). \end{aligned}$$

Since $\bar{v}_\mu \in E_{\lambda,a,\mu}$, we have

$$\begin{aligned} &\int_\Omega \nabla V_\mu^j.\nabla \frac{\partial \bar{v}_\mu}{\partial \lambda_i} = -\int_\Omega \nabla \frac{\partial V_\mu^j}{\partial \lambda_i}.\nabla \bar{v}_\mu = 0 \\ &\int_\Omega \nabla \frac{\partial V_\mu^j}{\partial \lambda_j}.\nabla \frac{\partial \bar{v}_\mu}{\partial \lambda_i} = -\int_\Omega \nabla \frac{\partial^2 V_\mu^j}{\partial \lambda_j \partial \lambda_i}.\nabla \bar{v}_\mu = O\left(\delta_{ij} \left|\nabla \frac{\partial^2 V_\mu^i}{\partial \lambda_i^2}\right|_2 |\nabla \bar{v}_\mu|_2\right) \\ &\int_\Omega \nabla \frac{\partial V_\mu^j}{\partial (a_j)_l}.\nabla \frac{\partial \bar{v}_\mu}{\partial \lambda_i} - \int_\Omega \nabla \frac{\partial^2 V_\mu^j}{\partial (a_j)_l \partial \lambda_i}.\nabla \bar{v}_\mu = O\left(\delta_{ij} \left|\nabla \frac{\partial^2 V_\mu^i}{\partial (a_i)_l \partial \lambda_i}\right|_2 |\nabla \bar{v}_\mu|_2\right). \end{aligned}$$

Then, (3.12,14) and estimates (C.8) yield

$$\frac{\partial \tilde{\tilde{K}}_\mu}{\partial \lambda_i}(\lambda, a) = \frac{\partial \tilde{K}_\mu}{\partial \lambda_i}(\bar{\alpha}_\mu, \lambda, a, \bar{v}_\mu) + O\left(\frac{1}{\lambda_i}(\frac{\mu^{1/4}}{|\lambda|^{3/2}} + \frac{\mu}{|\lambda|^2})\right)$$

whence, according to (3.17) and (C.20)

$$\frac{\partial \tilde{\tilde{K}}_\mu}{\partial \lambda_i}(\lambda, a) = -2\pi\frac{\mu^{1/2}}{\lambda_i^2} + o\left(\frac{\mu^{1/2}}{\lambda_i^{3/2}|\lambda|^{1/2}}\right). \tag{3.18}$$

On the other hand, Propositions 3.1 and 3.2 show that, necessarily

$$\tilde{v}_\mu = \bar{v}_\mu(\tilde{\alpha}_\mu, \tilde{\lambda}_\mu, \tilde{a}_\mu) \qquad \tilde{\alpha}_\mu = \bar{\alpha}_\mu(\tilde{\lambda}_\mu, \tilde{a}_\mu)$$

for μ large enough where, according to Subsection 3.1, $(\tilde{\alpha}_\mu, \tilde{\lambda}_\mu, \tilde{a}_\mu, \tilde{v}_\mu) = \Phi(u_\mu)$. Moreover, u_μ being a critical point of J_μ, $(\tilde{\lambda}_\mu, \tilde{a}_\mu)$ satisfies

$$\frac{\partial \tilde{\tilde{K}}_\mu}{\partial \lambda_i}(\tilde{\lambda}_\mu, \tilde{a}_\mu) = 0 \qquad 1 \leq i \leq k. \tag{3.19}$$

However, up to a subsequence and a reindexation, we may assume, without loss of generality, that $\tilde{\lambda}_\mu^1 = \min_{1\leq i\leq k}\tilde{\lambda}_\mu^i$ for μ large enough. Then, (3.18) implies that

$$\frac{\partial \tilde{\tilde{K}}_\mu}{\partial \lambda_1}(\tilde{\lambda}_\mu, \tilde{a}_\mu) = -2\pi\frac{\mu^{1/2}}{(\tilde{\lambda}_\mu^1)^2} + o\left(\frac{\mu^{1/2}}{(\tilde{\lambda}_\mu^1)^2}\right)$$

in contradiction with (3.19). This completes the proof of the theorem.

Remarks.

(1) If a_i were on the boundary of Ω, some additional term would occur in (3.18), involving the mean curvature of the frontier at a_i, and (3.19) would not lead to a contradiction - see [40].

(2) If, instead of (P_μ), we consider the subcritical problem

$$-\Delta u + \mu u = u^{5-\varepsilon_\mu}\ ,\ u > 0 \text{ in } \Omega\ ;\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

$\varepsilon_\mu > 0$, $\varepsilon_\mu \ln \mu \to 0$ as $\mu \to \infty$, and the corresponding modified functional, an additional term would occur in (3.18) which, up to a strictly positive constant, would be equal to $\varepsilon_\mu/\lambda_i$ - see [9, 39]. Then, the derivative of $\tilde{\tilde{K}}_\mu$ with respect to λ_i would vanish for some $\lambda_i \sim \mu^{1/2}/\varepsilon_\mu$. Therefore, the obstruction to the fulfillment of (3.19) disappears in the subcritical case, in accordance with the known results.

(3) In [14] are considered problems as

$$-\Delta u + \mu u = u^{(n+2)/(n-2)} + a(x)u^q\ ,\ u > 0 \text{ in } \Omega\ ;\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

$\Omega \in \mathbb{R}^n$, $n \geq 3$, $1 < q < (n+2)/(n-2)$. We notice that the additional term with respect to (P_μ) would introduce in (3.18) a quantity as $a(a_i)/\lambda_i^{(q+3)/2}$ if $q < 2$, $a(a_i)(\ln \lambda_i)/\lambda_i^{5/2}$ if $q = 2$, $a(a_i)/\lambda_i^{(7-q)/2}$ if $q > 2$. Then, we see that (3.19) could be satisfied for $a(a_i) > 0$ and $q > 3$. This agrees with Theorem 1.1 in [14].

(4) In the special case $k = 1$, the result could be easily derived from [40]. Namely, considering the functional

$$I_\mu(u) = \frac{\int_\Omega (|\nabla u|^2 + \mu u^2)}{\left(\int_\Omega u^6\right)^{1/3}}$$

[40] provides us with the following expansion for the equivalent of $\tilde{\tilde{K}}_\mu$, that we denote in the same way :

$$\frac{\partial \tilde{\tilde{K}}_\mu}{\partial \lambda}(\lambda, a) = 2\pi^{1/3}\left(-2\frac{\mu^{1/2}}{\lambda^2} + \frac{H(y)}{\lambda^2}(\ln \frac{\lambda}{2\mu^{1/2}} - \gamma - \frac{1}{2})\right)$$
$$+ O\left(\frac{1}{\lambda^2 \mu^{1/2}} + \frac{\mu}{\lambda^3} + \frac{\mu^{1/2}}{\lambda^3} \ln \frac{\lambda}{\mu^{1/2}}\right)$$

for a concentration point a on the boundary. If a lies in the interior of the domain, similar and easier computations would have given

$$\frac{\partial \tilde{\tilde{K}}_\mu}{\partial \lambda}(\lambda, a) = -4\pi^{1/3}\frac{\mu^{1/2}}{\lambda^2} + +O\left(\frac{1}{\lambda^2 \mu^{1/2}} \frac{\mu}{\lambda^3}\right)$$

which cannot vanish for large λ.

For $n \geq 5$, computations in [38] provide us with the expansion

$$\frac{\partial \tilde{\tilde{K}}_\mu}{\partial \lambda_i}(\lambda, a) = C_n \frac{H(a_i)}{\lambda_i^2} - C'_n \frac{\mu}{\lambda_i^3} + \text{lower order terms}$$

(with C_n and C'_n strictly positive constant) when the concentrations points are on the boundary. For interior points we would have

$$\frac{\partial \tilde{\tilde{K}}_\mu}{\partial \lambda_i}(\lambda, a) = -2C'_n \frac{\mu}{\lambda_i^3} + \text{lower order terms.}$$

Again, these quantities cannot vanish for large λ_i's, whence the equivalent of the Theorem under the assumption that the concentration points stay far from each other - as in [14].

Treating the case $n = 4$ in the same way would require, in order to obtain convenient expansions, to consider suitable approximate solutions, as we did in the case $n = 3$.

APPENDIX

A Proof of Proposition 2.1

The proof relies on the following lemma :

Lemma A.1 *Let $(u_\mu)_{\mu\geq\mu_0}$ be a sequence of nonconstant solutions to (P_μ). Let $\varepsilon>0$, $R>1$. There exists a constant $C_0=C_0(R)$ such that, for μ large enough and any compact set K*

$$\max_{x\in\bar\Omega\setminus K} d(x,K)^{1/2}u_\mu(x)\geq C_0$$

(with $d(x,K)=1$ if $K=\emptyset$) implies the existence of x_μ, local maximum point of u_μ in $\bar\Omega$, such that

$$(i)\quad \frac{\mu^{1/4}}{u_\mu(x_\mu)}<\varepsilon$$

$$(ii)\quad \left\|\frac{1}{u_\mu(x_\mu)}u_\mu\Big(\frac{x}{u_\mu^2(x_\mu)}+x_\mu\Big)-\frac{1}{(1+\frac{|x|^2}{3})^{1/2}}\right\|_{C^2\left(B(0,2R)\cap\Omega_\mu\right)}<\varepsilon$$

$$\textit{with } \Omega_\mu=u_\mu^2(x_\mu)(\Omega-x_\mu)$$

$$(iii)\quad u_\mu^2(x_\mu)d(x_\mu,K)>R$$

$$x_\mu\in\partial\Omega \qquad \textit{or} \qquad u_\mu^2(x_\mu)d(x_\mu,\partial\Omega)>\frac{1}{\varepsilon}.$$

Proof of Proposition 2.1. Lemma A.1, applied with $K=\emptyset$, provides us with $x_\mu=x_\mu^1$ such that (i) (ii) (iii) of Proposition 3.1 are satisfied. Then, we set

$$K=\bar B(x_\mu^1,r_\mu^1)\qquad \text{with } r_\mu^1=\frac{R}{u_\mu^2(x_\mu^1)}.$$

If $\max_{x\in\bar\Omega\setminus K}d(x,K)^{1/2}u_\mu(x)<C_0$, there is nothing more to prove. If not, Lemma A.1 provides us with a new $x_\mu=x_\mu^2$ such that (i) (ii) (iii) of Proposition 2.1 are again satisfied, and $\bar B(x_\mu^1,r_\mu^1)\cap\bar B(x_\mu^2,r_\mu^2)=\emptyset$, with $r_\mu^2=\frac{R}{u_\mu^2(x_\mu^2)}$. The process must stop after a finite number N_μ of steps since, because of (ii)

$$\int_{B(x_\mu^i,r_\mu^i)}|\nabla u_\mu|^2\geq\tau>0$$

with τ a constant which does not depend on i, μ. (Note that if (u_μ) is assumed to be bounded in $H^1(\Omega)$, N_μ is also bounded as μ goes to infinity.) We have

$$\max_{x\in\bar\Omega\setminus\cup_{i=1}^{N_\mu}B(x_\mu^i,r_\mu^i)}\Big(d\big(x,\cup_{i=1}^{N_\mu}B(x_\mu^i,r_\mu^i)\big)\Big)^{1/2}u_\mu(x)<C_0.$$

Consequently, if $x \notin \cup_i B(x^i_\mu, 2r^i_\mu)$

$$\Big(d\big(x, \{x^i_\mu,\ 1 \le i \le N_\mu\}\big)\Big)^{1/2} u_\mu(x) \le \Big(2d\big(x, \cup_i B(x^i_\mu, r^i_\mu)\big)\Big)^{1/2} u_\mu(x) < \sqrt{2} C_0$$

and, if $x \in \cup_i B(x^i_\mu, 2r^i_\mu)$, using (ii)

$$\begin{aligned}
&\Big(d\big(x, \{x^i_\mu, 1 \le i \le N_\mu\}\big)\Big)^{1/2} u_\mu(x) \\
&\qquad \le |x - x^i_\mu|^{1/2} u_\mu(x) \\
&\qquad \le |x - x^i_\mu|^{1/2} u_\mu(x) \Big(\big(1 + \frac{1}{3} u^4_\mu(x^i_\mu)|x - x^i_\mu|^2\big)^{-1/2} + \varepsilon\Big) \\
&\qquad \le 2^{-1/2} 3^{1/4} + 2^{1/2} \varepsilon R^{1/2}.
\end{aligned}$$

Therefore, (v) is satisfied and the proof of Proposition 2.1 is complete.

Proof of Lemma A.1. Arguing by contradiction, we assume that there exist a sequence (u_μ) of nonconstant solutions (P_μ) and a sequence (K_μ) of compact sets such that

$$\max_{x \in \bar\Omega \setminus K_\mu} d(x, K_\mu)^{1/2} u_\mu(x) \to \infty \qquad (\text{with } d(x, K_\mu) = 1 \text{ if } K_\mu = \emptyset)$$

and there is no x_μ, as specified in the lemma. Let $\tilde x_\mu \in \Omega \setminus K_\mu$ be a global maximum point of $d(x, K_\mu)^{1/2} u_\mu(x)$ in $\bar\Omega \setminus K_\mu$. We set

$$v_\mu(x) = \frac{1}{u_\mu(\tilde x_\mu)} u_\mu\Big(\frac{x}{u^2_\mu(\tilde x_\mu)} + \tilde x_\mu\Big) \qquad x \in \tilde\Omega_\mu = u^2_\mu(\tilde x_\mu)(\Omega - \tilde x_\mu)$$

and

$$R_\mu = \frac{1}{4} d(\tilde x_\mu, K_\mu) u^2_\mu(\tilde x_\mu).$$

By assumption, $R_\mu \to \infty$ as $\mu \to \infty$. Since

$$d\Big(\frac{x}{u^2_\mu(\tilde x_\mu)} + \tilde x_\mu, K_\mu\Big) \ge \frac{1}{2} d(\tilde x_\mu, K_\mu) \qquad \forall x \in B(0, R_\mu)$$

we have, for $x \in B(0, R_\mu) \cap \tilde\Omega_\mu$

$$\Big(\frac{1}{2} d(\tilde x_\mu, K_\mu)\Big)^{1/2} v_\mu(x) \le \big(d(\tilde x_\mu, K_\mu)\big)^{1/2}$$

whence

$$v_\mu(x) \le \sqrt{2} \qquad \forall x \in B(0, R_\mu).$$

On the other hand, v_μ satisfies

$$-\Delta v_\mu + \frac{\mu}{u^4_\mu(\tilde x_\mu)} v_\mu = v^5_\mu\ ,\ v_\mu > 0 \text{ in } \tilde\Omega_\mu\ ; \quad \frac{\partial v_\mu}{\partial \nu} = 0 \text{ on } \partial\tilde\Omega_\mu.$$

Let us assume that $\mu u_\mu^{-4}(\tilde{x}_\mu)$ goes to infinity as μ goes to infinity. w_μ defined as

$$w_\mu = v_\mu(\frac{x}{\alpha_\mu}) \qquad x \in \alpha_\mu \tilde{\Omega}_\mu \quad \text{with} \quad \alpha_\mu = \frac{\mu^{1/2}}{u_\mu^2(\tilde{x}_\mu)}$$

satisfies

$$-\Delta w_\mu + w_\mu = \frac{1}{\alpha_\mu^2} w_\mu^5 \ , \ w_\mu > 0 \text{ in } \alpha_\mu \tilde{\Omega}_\mu \ ; \quad \frac{\partial w_\mu}{\partial \nu} = 0 \text{ on } \partial(\alpha_\mu \tilde{\Omega}_\mu)$$

and

$$w_\mu(x) \leq \sqrt{2} \qquad \forall x \in B(0, \alpha_\mu R_\mu).$$

Elliptic theory shows that, up to a subsequence and a space rotation, w_μ converges in $C^2_{loc}(\mathbb{R}^3_T)$ to a limit w which satisfies

$$-\Delta w + w = 0 \ , \ 0 \leq w \leq \sqrt{2} \text{ in } \mathbb{R}^3_T \ ; \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial \mathbb{R}^3_T$$

with

$$\mathbb{R}^3_T = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3, \ x_3 > -T\} \qquad 0 \leq T \leq \infty.$$

This implies $w \equiv 0$, in contradiction with $w(0) = 1$. Therefore, $\mu u_\mu^{-4}(\tilde{x}_\mu)$ is bounded.

Up to a subsequence, we can assume

$$\mu u_\mu^{-4}(\tilde{x}_\mu) \to \theta \quad \text{ as } \mu \to \infty \qquad 0 \leq \theta < \infty$$

and, up to a space rotation, v_μ converges in $C^2_{loc}(\mathbb{R}^3_S)$ to a limit v which satisfies

$$-\Delta v + \theta v = v^5 \ , \ 0 \leq v \leq \sqrt{2} \text{ in } \mathbb{R}^3_S \ ; \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial \mathbb{R}^3_S.$$

Note that if $S < \infty$, v may be continued by reflection in a bounded solution of the equation in whole $\mathbb{R}^3$. As a consequence, if $\theta > 0$, v is constant, i.e. $v \equiv 0$ or $v = \theta^{1/4}$. Since $v(0) = 1$, $\theta = 1$ and $v \equiv 1$. This implies that u_μ is constant for μ large enough, in contradiction with the initial assumption.

Before proving this fact, let us complete the proof of the lemma. If $\theta = 0$, we know that v writes as - see [13]

$$v(x) = \frac{\lambda^{1/2}}{(1 + \lambda^2 \frac{|x - x_0|^2}{3})^{1/2}} \qquad \text{for some } x_0 \in \mathbb{R}^3, \ \lambda \in \mathbb{R}^*_+.$$

$v \leq \sqrt{2}$ and $v(0) = 1$ imply that

$$\lambda \leq \sqrt{2} \qquad \lambda^{1/2}(1 + \lambda^2 \frac{|x_0|^2}{3})^{-1/2} = 1$$

whence

$$1 \leq \lambda \leq \sqrt{2} \qquad |x_0|^2 = 3(\lambda - 1)/\lambda^2 \leq 3/4.$$

Moreover, if $S < \infty$, necessarily $x_0 \in \partial\mathbb{R}^3_S$. From the shape of v and the convergence of v_μ to v in $C^2_{loc}(\mathbb{R}^3_S)$, we deduce the existence, for large μ, of a local maximum point z_μ of v_μ, which goes to x_0 as μ goes to infinity. Then

$$x_\mu = \frac{z_\mu}{u_\mu^2(\tilde{x}_\mu)} + \tilde{x}_\mu$$

is a local maximum point of u_μ, and

$$\frac{u_\mu(x_\mu)}{u_\mu(\tilde{x}_\mu)} = v_\mu(z_\mu) \to \lambda^{1/2} \in [1, \sqrt{2}]$$

whence $\mu u_\mu^{-4}(\tilde{x}_\mu) \to 0$ as $\mu \to \infty$. Moreover, $|x_\mu - \tilde{x}_\mu| \leq \sqrt{3} u_\mu^{-2}(\tilde{x}_\mu)$ for μ large enough. As $d(\tilde{x}_\mu, K_\mu) u_\mu^2(\tilde{x}_\mu) \to \infty$, we know that for large μ, $x_\mu \in \bar{\Omega} \setminus K_\mu$, and $d(x_\mu, K_\mu) u_\mu^2(x_\mu) \to \infty$ as μ goes to infinity.

Then, we can repeat the same argument as previously, with x_μ instead of $\tilde{x}_\mu$. In this case 0, local maximum point of v_μ, is a critical point of v. As the only critical point of v is x_0, we obtain $x_0 = 0$, whence $S = 0$ or $S = \infty$, i.e.

$$u_\mu^2(x_\mu) d(x_\mu, \partial\Omega) \to 0 \quad \text{ or } \quad u_\mu^2(x_\mu) d(x_\mu, \partial\Omega) \to \infty.$$

$v(0) = 1$ gives $\lambda = 1$, and the convergence of v_μ to v in $C^2_{loc}(\mathbb{R}^3_S)$

$$\left\| \frac{1}{u_\mu(x_\mu)} u_\mu\left(\frac{x}{u_\mu^2(x_\mu)} + x_\mu\right) - \frac{1}{(1 + \frac{|x|^2}{3})^{1/2}} \right\|_{C^2\left(B(0,2R) \cap \Omega_\mu \cap \mathbb{R}^3_S\right)} = o(1) \tag{A.1}$$

for any $R > 0$, with $\Omega_\mu = u_\mu^2(x_\mu)(\Omega - x_\mu)$.

Actually, (A.1) holds in $C^2(B(0,2R) \cap \Omega_\mu)$. Indeed, if $S = 0$, up to a subsequence x_μ goes to some $\bar{x} \in \partial\Omega$ as μ goes to infinity. Up to a space rotation, we may assume that the tangent space to $\partial\Omega$ at $\bar{x}$ is parallel to the plane $x_3 = 0$. There exist $\mathcal{U}$ neigbourhood of $\bar{x}$ in $\mathbb{R}^3$, $r > 0$ and $\Psi : \mathcal{U} \to B(0,R) \subset \mathbb{R}^3$ a C^2-diffeomorphism such that

$$\Psi(\mathcal{U} \cap \Omega) = B^+(0,r) = \{x \in \mathbb{R}^3,\ |x| < r,\ x_3 > 0\}$$

and

$$\Psi(\bar{x}) = 0 \qquad \nabla\Psi(\bar{x}) = 0.$$

Then, for any r', $0 < r' < r$

$$\bar{v}_\mu(y) = \frac{1}{u_\mu(x_\mu)} u_\mu \left(\Psi^{-1}\left(\frac{y}{u_\mu^2(x_\mu)} + y_\mu\right) \right) \qquad \text{with } y_\mu = \Psi(x_\mu)$$

satisfies, for μ large enough

$$-\sum_{i,j} a_{ij}(y) \frac{\partial^2 \bar{v}_\mu}{\partial y_i \partial y_j} + \sum_i b_i(y) \frac{\bar{v}_\mu}{\partial y_i} + \frac{\mu}{u_\mu^4(x_\mu)} \bar{v}_\mu = v_\mu^5$$

in $B(0,r') \cap \{y_3 > -u_\mu^2(x_\mu)(y_\mu)_3\}$, and

$$\frac{\partial \bar{v}_\mu}{\partial y_3} = 0$$

on $B(0,r') \cap \{y_3 = -u_\mu^2(x_\mu)(y_\mu)_3\}$, with

$$a_{ij}(y) = \delta_{ij} + o(1) \qquad b_i(y) = o(1) \qquad \text{uniformly as } \mu \to \infty.$$

$\bar{v}_\mu$, continued by reflection in $B(0,r'/2)$, satisfies in $B(0,r'/2)$ the same equation. The same arguments as previously show that the continuation of $\bar{v}_\mu$ converges in $C^2_{loc}(\mathbb{R}^3)$ to $v(y) = (1+|y|^2/3)^{-1/2}$. Consequently, (A.1) holds in $C^2(B(0,2R) \cap \Omega_\mu)$, as announced.

Finally, we show that if $S = 0$, $x_\mu \in \partial\Omega$ for μ large enough. Indeed, assume that $x_\mu \notin \partial\Omega$. Since x_μ is a local maximum of v_μ, $y_\mu = \Psi(x_\mu)$ is a local maximum of $\bar{v}_\mu$, whence $\nabla \bar{v}_\mu(y_\mu) = 0$. Let y'_μ be the orthogonal projection of y_μ on the plane $x_3 = 0$. We have

$$\frac{\partial \bar{v}_\mu}{\partial y_3}(y_\mu) = \frac{\partial \bar{v}_\mu}{\partial y_3}(y'_\mu) = 0$$

Consequently, there exist $y''_\mu = ty_\mu + (1-t)y'_\mu$, $0 < t < 1$, such that $\frac{\partial^2 \bar{v}_\mu}{\partial y_3^2}(y''_\mu) \geq 0$. On the other hand

$$\frac{\partial^2 \bar{v}_\mu}{\partial y_3^2}(y''_\mu) \to \frac{\partial^2 v}{\partial y_3^2}(0) < 0$$

according to the shape of v, whence a contradiction.

To complete the proof of the lemma, it only remains to prove that

$$\theta_\mu = \mu u_\mu^{-4}(\tilde{x}_\mu) \to 1 \qquad \text{as } \mu \to \infty$$

implies that $u_\mu \equiv \mu^{1/4}\theta_\mu^{-1/4}$ for large μ, in contradiction with the assumption that u_μ is nonconstant. Let a_μ be a point in the closure of $B(0,R_\mu/2) \cap \tilde{\Omega}_\mu$, such that

$$|v_\mu(a_\mu) - \theta_\mu^{1/4}| = \|v_\mu - \theta_\mu^{1/4}\|_{L^\infty(B(0,R_\mu/2)\cap\tilde{\Omega}_\mu)}.$$

$\tilde{v}_\mu$ defined as $\tilde{v}_\mu = v_\mu(x + a_\mu)$, satisfies

$$-\Delta \tilde{v}_\mu + \theta_\mu \tilde{v}_\mu = \tilde{v}_\mu^5 \ , \ \tilde{v}_\mu > 0 \text{ in } \tilde{\Omega}_\mu - a_\mu \ ; \quad \frac{\partial \tilde{v}_\mu}{\partial \nu} = 0 \text{ on } \partial(\tilde{\Omega}_\mu - a_\mu)$$

and

$$\tilde{v}_\mu(x) \leq \sqrt{2} \qquad \forall x \in B(0,R_\mu/2) \cap (\tilde{\Omega}_\mu - a_\mu)$$

since

$$\Big(B(0,R_\mu/2) \cap (\tilde{\Omega}_\mu - a_\mu)\Big) \subset \Big((B(0,R_\mu) \cap (\tilde{\Omega}_\mu - a_\mu)) - a_\mu\Big).$$

Up to a subsequence and a space rotation, $\tilde{v}_\mu$ converges in $C^2_{loc}(\mathbb{R}^3_{T'})$, $0 \le T' \le \infty$, to a limit $\tilde{v}$ which satisfies

$$-\Delta\tilde{v} + \tilde{v} = \tilde{v}^5 \ , \ 0 \le \tilde{v} \le \sqrt{2} \text{ in } \mathbb{R}^3_{T'} \ ; \quad \frac{\partial \tilde{v}}{\partial \nu} = 0 \text{ on } \partial\mathbb{R}^3_{T'}.$$

As a consequence, $\tilde{v} \equiv 0$ or $\tilde{v} \equiv 1$. Therefore, $|v_\mu(a_\mu) - \theta_\mu^{1/4}| = \|v_\mu - \theta_\mu^{1/4}\|_{L^\infty(B(0,R_\mu/2)\cap\tilde{\Omega}_\mu)}$ goes to a limit l, $l = 1$ if $\tilde{v} \equiv 0$, and $l = 0$ if $\tilde{v} \equiv 1$.

Let us assume that $l = 1$. In this case, $v_\mu(a_\mu) = \tilde{v}_\mu(0)$ goes to zero as μ goes to infinity. We also know that $v_\mu(0) = 1$, whence the existence, for μ large enough, of $b_\mu \in (B(0, R_\mu/2) \cap \tilde{\Omega}_\mu)$ such that $v_\mu(b_\mu) = 1/2$. Setting $\tilde{v}'_\mu(x) = v_\mu(x + b_\mu)$, and repeating the same argument as above, we see that up to a subsequence and a space rotation, $\tilde{v}'_\mu$ converges in $C^2_{loc}(\mathbb{R}^3_{T''})$, $0 \le T'' \le \infty$, to a limit $\tilde{v}'$. As previously, $\tilde{v}'$ has to be identically 0 or 1, in contradiction with $\tilde{v}'(0) = 1/2$. Therefore, $l = 0$.

Finally, let us assume that $v_\mu \not\equiv \theta_\mu^{1/4}$ in $B(0, R_\mu/2) \cap \tilde{\Omega}_\mu$. We set

$$\tilde{\tilde{v}}_\mu(x) = \frac{\tilde{v}_\mu(x) - \theta_\mu^{1/4}}{\tilde{v}_\mu(0) - \theta_\mu^{1/4}}$$

which satisfies

$$-\Delta\tilde{\tilde{v}}_\mu + \theta_\mu\tilde{\tilde{v}}_\mu = \tilde{\tilde{v}}_\mu(\theta_\mu^4 + \theta_\mu^3\tilde{v}_\mu + \theta_\mu^2\tilde{v}_\mu^2 + \theta_\mu\tilde{v}_\mu^3 + \tilde{v}_\mu^4) \quad \text{in } \ \tilde{\Omega}_\mu - a_\mu$$

and

$$|\tilde{\tilde{v}}_\mu| \le 1 \quad \text{ in } B(0, R_\mu/2) \cap (\tilde{\Omega}_\mu - a_\mu).$$

Therefore, up to a subsequence, $\tilde{\tilde{v}}_\mu$ converges in $C^2_{loc}(\mathbb{R}^3_{T'})$ to a limit $\tilde{\tilde{v}}$ which satisfies

$$-\Delta\tilde{\tilde{v}} = 4\tilde{\tilde{v}} \text{ in } \mathbb{R}^3_{T'} \ ; \quad \frac{\partial \tilde{\tilde{v}}}{\partial \nu} = 0 \text{ on } \partial\mathbb{R}^3_{T'}$$

and $\|\tilde{\tilde{v}}_\mu\|_{L^\infty(\mathbb{R}^3_{T"})} \le 1$. It follows that $\tilde{\tilde{v}} \equiv 0$, in contradiction with $|\tilde{\tilde{v}}(0)| = 1$. Therefore, v_μ is constant in $B(0, R_\mu/2) \cap \tilde{\Omega}_\mu$ and, actually, in whole $\tilde{\Omega}_\mu$. This means that u_μ is constant in Ω, a contradiction.

B Proof of Proposition 2.5

Let u_μ be a sequence, bounded in $H^1(\Omega)$, of solutions to (P_μ). On one hand, we have

$$\int_\Omega (|\nabla u_\mu|^2 + \mu u_\mu^2) = \int_\Omega u_\mu^6. \tag{B.1}$$

On the other hand, the continuous embedding of $H^1(\Omega)$ into $L^6(\Omega)$ yields

$$\left(\int_\Omega u_\mu^6\right)^{2/3} \le C\left(\int_\Omega (|\nabla u_\mu|^2 + \mu u_\mu^2)\right).$$

Therefore, $\|u_\mu\|_{H^1(\Omega)} \geq C^{-3/2}$, and $|u_\mu|_6^6 \geq C^{-3}$. Passing to a subsequence, we may assume that $|u_\mu|_6^6 \to l > 0$ as $\mu \to \infty$. We set

$$Q_\mu(t) = \sup_{x\in\Omega} \int_{(x+tB(0,1))\cap\Omega} u_\mu^6 \qquad t > 0.$$

Q_μ is continuous and increasing. Let R be such that $\Omega \subset x_0 + B(0,R)$, for some $x_0 \in \Omega$. Consequently, $Q_\mu(R) = |u_\mu|_6^6 = l + o(1)$. Choosing $\rho > 0$ such that

$$\rho < \min(\frac{l}{2}, A^{3/2}) \tag{B.2}$$

with

$$A = \inf_{u\in H^1(\Omega)} \frac{\|u_\mu\|_{H^1(\Omega)}}{|u_\mu|_6^2}$$

for large μ there exist ε_μ, $0 < \varepsilon_\mu < R$, and $a_\mu \in \bar{\Omega}$ such that

$$Q_\mu(\varepsilon_\mu) = \int_{(a_\mu+\varepsilon_\mu B(0,1))\cap\Omega} u_\mu^6 = \rho. \tag{B.3}$$

We set

$$\tilde{u}_\mu(x) = \varepsilon_\mu^{1/2} u_\mu(\varepsilon_\mu x + a_\mu) \qquad x \in \Omega_\mu = (\Omega - a_\mu)/\varepsilon_\mu$$

which satisfies

$$-\Delta \tilde{u}_\mu + \mu\varepsilon_\mu^2 \tilde{u}_\mu = \tilde{u}_\mu^5 \ , \ \tilde{u}_\mu > 0 \text{ in } \Omega_\mu \ ; \quad \frac{\partial \tilde{u}_\mu}{\partial \nu} = 0 \text{ on } \partial\Omega_\mu. \tag{B.4}$$

Up to some subsequence, one of the three cases occur :

$$\begin{aligned} &(i) \quad \mu\varepsilon_\mu^2 \to \infty \\ &(ii) \quad \mu\varepsilon_\mu^2 \to \alpha > 0 \\ &(iii) \quad \mu\varepsilon_\mu^2 \to 0. \end{aligned}$$

We are going to prove that, actually, (iii) is the only possible case.

Let us assume, first, that $\mu\varepsilon_\mu^2$ goes to infinity. (B.1) and the boundedness of (u_μ) in $H^1(\Omega)$ implies that $\mu|u_\mu|_2^2$ is bounded, whence

$$\int_{\Omega_\mu} \tilde{u}_\mu^2 = \frac{1}{\mu\varepsilon_\mu^2} \int_\Omega u_\mu^2 \to 0 \quad \text{ as } \mu \to \infty. \tag{B.5}$$

Let ζ be a smooth positive function from $\mathbb{R}^3$ to $\mathbb{R}$, with $S = supp\zeta \subset b + B(0,1)$ for some $b \in \mathbb{R}^3$. We have

$$\int_{\Omega_\mu} \nabla\tilde{u}_\mu . \nabla(\zeta^2\tilde{u}_\mu) + \mu\varepsilon_\mu^2 \int_{\Omega_\mu} \tilde{u}_\mu(\zeta^2\tilde{u}_\mu) = \int_{\Omega_\mu} \tilde{u}_\mu^5(\zeta^2\tilde{u}_\mu).$$

Therefore

$$\int_{\Omega_\mu} |\nabla(\zeta\tilde{u}_\mu)|^2 - \int_{\Omega_\mu} |\nabla\zeta|^2\tilde{u}_\mu^2 + \mu\varepsilon_\mu^2 \int_{\Omega_\mu} (\zeta\tilde{u}_\mu)^2 = \int_{\Omega_\mu} \tilde{u}_\mu^4(\zeta\tilde{u}_\mu)^2.$$

Noticing that

$$\inf_{v\in H^1(\Omega_\mu)} \frac{|\nabla v|_2^2+\varepsilon_\mu^2|v|_2^2}{|v|_6^2} = \inf_{u\in H^1(\Omega)} \frac{\|u\|_{H^1}}{|u|_6^2} = A \tag{B.6}$$

taking account of (B.5) and using Hölder inequality, we obtain

$$A\left(\int_{\Omega_\mu}(\zeta\tilde{u}_\mu)^6\right)^{1/3} \le \left(\int_{S\cap\Omega_\mu}\tilde{u}_\mu^6\right)^{2/3}\left(\int_{\Omega_\mu}(\zeta\tilde{u}_\mu)^6\right)^{1/3} + o(1).$$

In view of (B.3-4)

$$\int_{S\cap\Omega_\mu}\tilde{u}_\mu^6 \le \int_{(a_\mu+\varepsilon_\mu b+\varepsilon_\mu B(0,1))\cap\Omega} u_\mu^6 \le Q_\mu(\varepsilon_\mu) < A^{3/2}$$

so that, for any smooth ζ such that $S = supp\zeta \subset b + B(0,1)$, $b \in \mathbb{R}^3$

$$\int_{\Omega_\mu}(\zeta\tilde{u}_\mu)^6 \to 0 \quad \text{as } \mu\to\infty \tag{B.7}$$

in contradiction with (B.3).

Let us assume now that $\mu\varepsilon_\mu^2$ is bounded. In particular, ε_μ goes to zero and, up to a subsequence and a space rotation, Ω_μ goes to $\mathbb{R}^3_T$, with

$$\mathbb{R}^3_T = \left\{x=(x_1,x_2,x_3)\in\mathbb{R}^3,\ x_3>-T\right\} \qquad T=\lim_{\mu\to\infty} d(a_\mu,\partial\Omega)/\varepsilon_\mu.$$

Noticing that $|\tilde{u}_\mu|_{L^6(\Omega_\mu)} = |u_\mu|_{L^6(\Omega)}$ and $|\nabla\tilde{u}_\mu|_{L^2(\Omega_\mu)} = |\nabla u_\mu|_{L^2(\Omega)}$, we may assume that there exist $\tilde{u}\in L^6(\mathbb{R}^3_T)$, $\tilde{v}\in L^2(\mathbb{R}^3_T)$ such that for any compact set $K\in\mathbb{R}^3_T$ ($K\in\Omega_\mu$ for μ large enough)

$$\tilde{u}_\mu \rightharpoonup \tilde{u} \text{ in } L^6(K) \qquad \nabla\tilde{u}_\mu \rightharpoonup \tilde{v} \text{ in } L^2(K).$$

Moreover, for any $\varphi\in C_0^\infty(\mathbb{R}^3_T)$

$$\int_{\Omega_\mu}\nabla\tilde{u}_\mu.\nabla\varphi \to \int_{\mathbb{R}^3_T}\tilde{v}.\nabla\varphi \quad \text{and} \quad \int_{\Omega_\mu}\nabla\tilde{u}_\mu.\nabla\varphi \to -\int_{\mathbb{R}^3_T}\tilde{u}\Delta\varphi$$

whence $\tilde{v}=\nabla\tilde{u}$.

Assuming that case (ii) occurs, $\tilde{u}$ satisfies

$$-\Delta\tilde{u}+\alpha\tilde{u} = \tilde{u}^5 \quad \text{in } \mathbb{R}^3_T. \tag{B.8}$$

Moreover, $\tilde{u}\ge 0$, since, along some subsequence, $\tilde{u}_\mu\to\tilde{u}$ almost everywhere.

If $T=\infty$, $\mathbb{R}^3_T=\mathbb{R}^3$, and the only solutions to (B.8) are $\tilde{u}\equiv 0$ or $\tilde{u}\equiv\alpha^{1/4}$. $\tilde{u}\equiv\alpha^{1/4}$ is excluded, since $\tilde{u}\in L^6(\mathbb{R}^3)$. If $\tilde{u}\equiv 0$, $\tilde{u}_\mu$ goes to zero in $L^2(K)$, for any compact set K. Then, using this result instead of (B.5), the same argument as previously leads to (B.7), that is a contradiction.

If $T < \infty$, we first notice that the normal derivative of $\tilde{u}$ vanishes on $\partial\mathbb{R}^3_T$. Indeed, from (B.4) we know that for any $\varphi \in C_0^\infty(\mathbb{R}^3)$

$$\int_{\Omega_\mu} \nabla\tilde{u}_\mu.\nabla\varphi + \mu\varepsilon_\mu^2 \int_{\Omega_\mu} \tilde{u}_\mu\varphi = \int_{\Omega_\mu} \tilde{u}_\mu^5\varphi$$

from which we easily deduce that

$$\int_{\mathbb{R}^3_T} \nabla\tilde{u}.\nabla\varphi + \alpha \int_{\mathbb{R}^3_T} \tilde{u}\varphi = \int_{\mathbb{R}^3_T} \tilde{u}^5\varphi$$

whence

$$\frac{\partial\tilde{u}}{\partial\nu} = 0 \quad \text{on } \partial\mathbb{R}^3_T.$$

Then, we can continue $\tilde{u}$ by reflection with respect to $x_3 = -T$, and the continuation, still denoted $\tilde{u}$, satisfies (B.8) in whole $\mathbb{R}^3$, whence $\tilde{u} \equiv 0$ or $\tilde{u} \equiv \alpha^{1/4}$. $\tilde{u} \equiv \alpha^{1/4}$ is again impossible, since $\tilde{u} \in L^6(\mathbb{R}^3)$. $\tilde{u} \equiv 0$ implies, as previously, that $\tilde{u}_\mu$ goes to zero in $L^2(K)$, for any compact set $K \in \mathbb{R}^3_T$.

If $\bar{B}(0,1) \in \mathbb{R}^3_T$ (i.e. $T > 1$), we can repeat the argument which leads to (B.7), hence a contradiction. If $0 \leq T \leq 1$, $\tilde{u}_\mu$ may be continued in $B(0,2) \setminus \Omega_\mu$ in such a way that, $\tilde{u}'_\mu$ denoting the continuation of $\tilde{u}_\mu$ in $\Omega_\mu \cup B(0,2)$

$$\|\tilde{u}'_\mu\|_{H^1(B(0,2))} \leq C\|\tilde{u}_\mu\|_{H^1(B(0,2)\cap\Omega_\mu)} \leq C'$$

C and C' independent of μ. Then, along some subsequence

$$\tilde{u}'_\mu \rightharpoonup \tilde{u}' \ \text{ in } H^1(B(0,2)) \qquad \tilde{u}'_\mu \to \tilde{u}' \ \text{ in } L^2(B(0,2))$$

with $\tilde{u}' = \tilde{u} = 0$ in $B(0,2) \cap \mathbb{R}^3_T$. Consequently

$$\int_{\Omega_\mu\cap B(0,2)} \tilde{u}_\mu^2 \leq 2\int_{\Omega_\mu\cap B(0,2)} (\tilde{u}_\mu - \tilde{u}'_\mu)^2 + 2\int_{\Omega_\mu\cap B(0,2)} \tilde{u'}^2_\mu \to 0 \text{ as } \mu \to \infty$$

and we are still able to prove (B.7), hence again a contradiction.

As announced, (iii) is the only possible case, and $\tilde{u}$ satisfies

$$-\Delta\tilde{u} = \tilde{u}^5 \ , \ \tilde{u} \geq 0 \text{ in } \mathbb{R}^3_T \ ; \quad \frac{\partial\tilde{u}}{\partial\nu} = 0 \text{ on } \partial\mathbb{R}^3_T \tag{B.9}$$

with $\nabla\tilde{u} \in L^2(\mathbb{R}^3_T)$, $\tilde{u} \in L^6(\mathbb{R}^3_T)$. If $\tilde{u} \equiv 0$, reasoning as previously provides us with a contradiction. Therefore, there exist $\lambda \in \mathbb{R}^*_+$ and $a \in \partial\mathbb{R}^3$ such that, according to [13]

$$\tilde{u}(x) = 3^{1/4}\frac{\lambda^{1/2}}{(1+\lambda^2|x-a|^2)^{1/2}}$$

and either $T = \infty$, or $T < \infty$ and $a \in \partial\mathbb{R}^3_T$. Then, we set

$$u^{(1)}_\mu(x) = u_\mu(x) - 3^{1/4}V_{\mu,\lambda/\varepsilon_\mu,a_\mu+\varepsilon_\mu a}(x) \qquad x \in \Omega \tag{B.10}$$

where V is defined by (2.13-14). We notice that

$$\frac{\mu}{(\lambda/\varepsilon_\mu)^2} \to 0 \tag{B.11}$$

and

$$(\lambda/\varepsilon_\mu)d(a_\mu+\varepsilon_\mu a,\partial\Omega) \to \infty \ (T=\infty) \text{ or } (\lambda/\varepsilon_\mu)d(a_\mu+\varepsilon_\mu a,\partial\Omega) \to 0 \ (T<\infty). \tag{B.12}$$

Indeed, $a \in \partial\mathbb{R}^3$ means that a is limit of points of $\partial\Omega_\mu$, so that we may write

$$a = \frac{x_\mu - a_\mu}{\varepsilon_\mu} + y_\mu \qquad x_\mu \in \partial\Omega \qquad y_\mu \to 0 \text{ in } \mathbb{R}^3$$

and

$$(\lambda/\varepsilon_\mu)d(a_\mu+\varepsilon_\mu a,\partial\Omega) \le \lambda|y_\mu| \to 0 \quad \text{ as } \mu\to\infty.$$

We claim that

$$\int_\Omega |\nabla u_\mu^{(1)}|^2 = \int_\Omega |\nabla u_\mu|^2 - \Big(S^{3/2} \text{ if } T=\infty \ ; \ S^{3/2}/2 \text{ if } T<\infty\Big) + o(1) \tag{B.13}$$

where

$$S = \inf_{u\in H_0^1(\Omega)} \frac{|\nabla u|_2^2}{|u|_6^2} = \frac{3\pi^{4/3}}{2^{4/3}}$$

is the Sobolev constant, and

$$-\Delta u_\mu^{(1)} + \mu u_\mu^{(1)} = (u_\mu^{(1)+})^5 + f_\mu \ \text{ in } \Omega \qquad u_\mu^{(1)+} = \max(u_\mu^{(1)},0) \tag{B.14}$$

with

$$(f_\mu,\varphi)_{H^{-1}(\Omega),H^1(\Omega)} = o(\|\varphi\|_{H^1(\Omega)}) \quad \text{uniformly for } \varphi\in H^1(\Omega)$$

as μ goes to infinity.

If $u_\mu^{(1)}$ does not go to zero in $H^1(\Omega)$, we can apply to $u_\mu^{(1)}$ the same arguments as those that we used concerning u_μ. It is easily checked that the presence of f_μ does not affect the situation; in the same way, (B.12) ensures that the normal derivative of $u_\mu^{(1)}$ on $\partial\Omega$ is sufficiently small for our purposes. In particular

$$\int_{\Omega_\mu} \left|\frac{\partial \tilde u_\mu^{(1)}}{\partial\nu}\right| \tilde u_\mu^{(1)} \to 0 \ \text{ and } \int_{\Omega_\mu} \frac{\partial \tilde u_\mu^{(1)}}{\partial\nu}\varphi \to 0$$

for any $\varphi\in C_0^\infty(\mathbb{R}^3)$, as $\mu\to\infty$. Then, we obtain some $u_\mu^{(2)}$ which either goes to zero in $H^1(\Omega)$, or may be treated as previously to define some $u_\mu^{(3)}$, and so on. The process must stop after a finite number of steps since, according to (B.13), $|\nabla u|_2^2$ loses each time some fixed amount. In the end, we obtain that u_μ writes as

$$u_\mu = 3^{1/4}\sum_{i=1}^k V_{\mu,\lambda_\mu^i,a_\mu^i} + v_\mu \qquad v_\mu \to 0 \text{ in } H^1(\Omega) \tag{B.15}$$

with

$$\mu^{1/2}/\lambda_\mu^i \to 0$$
$$\lambda_\mu^i d(a_\mu^i, \partial\Omega) \to 0 \quad \text{or} \quad \lambda_\mu^i d(a_\mu^i, \partial\Omega) \to \infty$$

as $\mu \to \infty$. Moreover, in view of (B.13), we have

$$|\nabla u_\mu|_2^2 = 3^{1/2} \sum_{i=1}^{k} |\nabla V_{\mu,\lambda_\mu^i,a_\mu^i}|_2^2 + o(1). \tag{B.16}$$

As, on the other hand, we deduce from computations in [7] that

$$\int_\Omega \nabla V_{\mu,\lambda_\mu^i,a_\mu^i} . \nabla V_{\mu,\lambda_\mu^j,a_\mu^j} \geq C \left(\frac{\lambda_\mu^i}{\lambda_\mu^j} + \frac{\lambda_\mu^j}{\lambda_\mu^i} + \lambda_\mu^i \lambda_\mu^j |a_\mu^i - a_\mu^j|^2 \right)^{1/2} \tag{B.17}$$

where C is a strictly positive constant, (B.15-16) imply that the right hand side in (B.17) goes to zero if $i \neq j$, and the proof of Proposition 2.5 is complete.

Proof of (B.13). In view of (B.10), we have

$$\int_\Omega |\nabla u_\mu^{(1)}|^2 = \int_\Omega |\nabla u_\mu|^2 - 2.3^{1/4} \int_\Omega \nabla u_\mu . \nabla V_\mu + 3^{1/2} \int_\Omega |\nabla V_\mu|^2$$

with $V_\mu = V_{\mu,\lambda/\varepsilon_\mu, a_\mu + \varepsilon_\mu a}$, for sake of simplicity. On one hand, it follows from (2.16) that

$$\int_\Omega |\nabla V_\mu|^2 = \int_\Omega |\nabla U_\mu|^2 + o(1)$$

with $U_\mu = U_{\mu,\lambda/\varepsilon_\mu, a_\mu + \varepsilon_\mu a}$, and standard computations yield

$$\int_\Omega |\nabla U_\mu|^2 = S^{3/2} + o(1) \quad \text{if} \quad \frac{1}{\varepsilon_\mu} d(a_\mu + \varepsilon_\mu a, \partial\Omega) \to \infty$$
$$\int_\Omega |\nabla U_\mu|^2 = \frac{S^{3/2}}{2} + o(1) \quad \text{if} \quad \frac{1}{\varepsilon_\mu} d(a_\mu + \varepsilon_\mu a, \partial\Omega) \to 0.$$

On the other hand, still using (2.16) and the boundedness of u_μ in $H^1(\Omega)$

$$\begin{aligned}
\int_\Omega \nabla u_\mu . \nabla V_\mu &= \int_\Omega \nabla u_\mu . \nabla U_\mu + o(1) \\
&= \int_{\Omega_\mu} \nabla \tilde{u}_\mu . \nabla \tilde{u} + o(1) \\
&= 3^{1/4} \int_{\mathbb{R}_T^3} |\nabla U_{\lambda,a}|^2 + o(1)
\end{aligned}$$

and

$$\int_{\mathbb{R}_T^3} |\nabla U_{\lambda,a}|^2 = S^{3/2} \text{ if } T = \infty \quad (\text{i.e. } \frac{1}{\varepsilon_\mu} d(a_\mu + \varepsilon_\mu a, \partial\Omega) \to \infty)$$
$$\int_{\mathbb{R}_T^3} |\nabla U_{\lambda,a}|^2 = \frac{S^{3/2}}{2} + o(1) \text{ if } T < \infty,\ a \in \partial\mathbb{R}_T^3 \quad (\frac{1}{\varepsilon_\mu} d(a_\mu + \varepsilon_\mu a, \partial\Omega) \to 0)$$

whence (B.13).

Proof of (B.14). From (B.10) and (2.15) we know that

$$-\Delta u_\mu^{(1)} + \mu u_\mu^{(1)} = u_\mu^5 - (3^{1/4}V_\mu)^5 + \mu 3^{1/4}\left(U_\mu - \frac{1}{(\lambda/\varepsilon_\mu)^{1/2}|x - a_\mu - \varepsilon_\mu a|}\right)$$

in Ω. Let $\varphi \in H^1(\Omega)$. We notice that

$$\begin{aligned}\mu \int_\Omega \Big(U_\mu - &\frac{1}{(\lambda/\varepsilon_\mu)^{1/2}|x - a_\mu - \varepsilon_\mu a|}\Big)\varphi \\ &\leq \mu|\varphi|_2 \left(\int_\Omega \Big(U_\mu - \frac{1}{(\lambda/\varepsilon_\mu)^{1/2}|x - a_\mu - \varepsilon_\mu a|}\Big)^2\right)^{1/2} \\ &\leq C\mu\varepsilon_\mu^2|\varphi|_2\end{aligned}$$

as a direct computation shows. Secondly, setting $\varphi_\mu = \varphi_{\mu,\lambda/\varepsilon_\mu,a_\mu+\varepsilon_\mu a}$, we have in view of (2.16)

$$\begin{aligned}\int_\Omega (V_\mu^5 - U_\mu^5)\varphi \leq &\int_\Omega (U_\mu^4\varphi_\mu + \varphi_\mu^5)|\varphi| \\ &\leq C'|\varphi_\mu|_6\|\varphi\|_{H^1(\Omega)}\end{aligned}$$

using Hölder inequality, the embedding of $H^1(\Omega)$ in $L^6(\Omega)$, the boundedness of (U_μ) in $L^6(\Omega)$ and the fact that φ_μ goes to zero in $L^6(\Omega)$ as $\mu\varepsilon_\mu^2$ goes to zero. Consequently, we are left to show that

$$\left|\int_\Omega \left((u_\mu^{(1)+})^5 - u_\mu^5 + (3^{1/4}U_\mu)^5\right)\varphi\right| \leq C_\mu\|\varphi\|_{H^1(\Omega)} \tag{B.18}$$

with C_μ independent of φ, $C_\mu \to 0$ as $\mu \to \infty$. We proceed as in [15]. Let I_μ denote the left hand side integral in (B.18). Setting $\bar U_\mu = 3^{1/4}U_\mu$, we have

$$\begin{aligned}I_\mu &= \int_\Omega \varphi \int_0^{\bar U_\mu} \frac{\partial}{\partial t}\left[\left((u_\mu - t)^+\right)^5 + t^5\right] dt \\ &= -5\int_\Omega \varphi \int_0^{\bar U_\mu} \left[\left((u_\mu - t)^+\right)^4 + t^4\right] dt \\ &= \int_0^1 \left(\int_\Omega \left[\left((u_\mu - s\bar U_\mu)^+\right)^4 - (1-s)^4\bar U_\mu^4\right]\bar U_\mu\varphi\right) ds.\end{aligned}$$

Therefore

$$\begin{aligned}|I_\mu| &\leq 5|\varphi|_6 \max_{0\leq s\leq 1}\int_\Omega \left|\left((u_\mu - s\bar U_\mu)^+\right)^4 - (1-s)^4\bar U_\mu^4\right|^{6/5}\bar U_\mu^{6/5} \\ &\leq C|\varphi|_{H^1(\Omega_\mu)}\int_\Omega \left|\left((\tilde u_\mu - s\tilde u)^+\right)^4 - (1-s)^4\tilde u^4\right|^{6/5}\tilde u^{6/5}.\end{aligned}$$

We know that $\|\tilde u_\mu\|_{H^1(\Omega)}$ is bounded, and that $\tilde u_\mu \rightharpoonup \tilde u$ in $H^1(K)$, $\tilde u_\mu \to \tilde u$ in $L^q(\Omega)$, $1 \leq q < 6$, for any compact set $K \in \mathbb{R}^3$. The conclusion follows easily.

C Estimates

In this last part, we collect the integral estimates which are used in Section 3. First, we recall that in [40] the following is proved, for $V = V_{\mu,\lambda,a}$

$$\int_\Omega |\nabla V|^2 = \frac{3\pi^2}{4} - 6\pi\frac{\mu^{1/2}}{\lambda} + O(\frac{1}{\lambda\mu^{1/2}}) \tag{C.1}$$

$$\int_\Omega V^2 = \frac{2\pi}{\lambda\mu^{1/2}} + O(\frac{1}{\lambda\mu^{3/2}}) \tag{C.2}$$

$$\int_\Omega V^6 = \frac{\pi^2}{4} - 8\pi\frac{\mu^{1/2}}{\lambda} + O(\frac{\mu^{1/2}}{\lambda^2}) \tag{C.3}$$

as a does not approach the boundary of Ω, and $\mu^{1/2}/\lambda$ goes to infinity. Actually, a is assumed in [40] to be on $\partial\Omega$. The above results follow from the same computations, made easier by the fact that a boundary effect has not to be considered. In the same way, we have

$$\int_\Omega \nabla V.\nabla\frac{\partial V}{\partial\lambda} = 3\pi\frac{\mu^{1/2}}{\lambda^2} + O(\frac{1}{\lambda^2\mu^{1/2}}) \qquad \int_\Omega \nabla V.\nabla\frac{\partial V}{\partial a_j} = O(\mu^{1/2}) \tag{C.4}$$

$$\int_\Omega V\frac{\partial V}{\partial\lambda} = -\frac{\pi}{\lambda^2\mu^{1/2}} + O(\frac{1}{\lambda^2\mu^{3/2}}) \qquad \int_\Omega V\frac{\partial V}{\partial a_j} = O(\frac{1}{\mu^{1/2}}) \tag{C.5}$$

$$\int_\Omega V^5\frac{\partial V}{\partial\lambda} = \frac{4\pi\mu^{1/2}}{3\lambda^2} + O(\frac{\mu^{1/2}}{\lambda^3}) \qquad \int_\Omega V^5\frac{\partial V}{\partial a_j} = O(\mu^{1/2}) \tag{C.6}$$

and

$$\int_\Omega \left|\nabla\frac{\partial V}{\partial\lambda}\right|^2 = \frac{15\pi^2}{64\lambda^2} + O(\frac{\mu^{1/2}}{\lambda^3}) \qquad \int_\Omega \left|\nabla\frac{\partial V}{\partial a_j}\right|^2 = \frac{15\pi^2}{64}\lambda^2 + O(\lambda\mu^{1/2}) \tag{C.7}$$

$$\int_\Omega \left|\nabla\frac{\partial^2 V}{\partial\lambda^2}\right|^2 = O(\frac{1}{\lambda^2}) \qquad \int_\Omega \left|\nabla\frac{\partial^2 V}{\partial\lambda\partial a_j}\right|^2 = O(1). \tag{C.8}$$

In view of (3.13), we have also to estimate integrals in which both V_i and V_j occur, $i \neq j$. Let us prove, for example

$$\int_\Omega \nabla V_i.\nabla V_j = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}) \tag{C.9}$$

as $\mu^{1/2}/\lambda_i$, $\mu^{1/2}/\lambda_j$ go to infinity, a_i and a_j do not approach the boundary of Ω and $|a_i - a_j| > \gamma/2\mu^{1/2}$. From (2.15) we deduce that

$$\int_\Omega \nabla V_i.\nabla V_j = \int_{\partial\Omega}\frac{\partial V_i}{\partial\nu}V_j + \int_\Omega\left(3U_i^5 - \frac{\mu e^{-\mu^{1/2}|x-a_i|}}{\lambda^{1/2}|x-a_i|}\right)V_j. \tag{C.10}$$

It follows from the definition (2.13-14) of V that, a_i and a_j staying far from $\partial\Omega$

$$V_j = o(\frac{1}{\lambda_j^{1/2}}) \qquad |\nabla V_i| = o(\frac{\mu^{1/2}}{\lambda_i^{1/2}}) \qquad \text{on } \partial\Omega$$

whence

$$\int_{\partial\Omega} \frac{\partial V_i}{\partial \nu} V_j = o(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}). \tag{C.11}$$

Turning now to the integral on Ω, we notice that

$$V_j = O(\frac{\mu^{1/2}}{\lambda_j^{1/2}}) \qquad \text{outside of } B_j = B(a_j, \gamma/4\mu^{1/2}). \tag{C.12}$$

On the other hand, taking $R' > 0$ such that $\Omega \subset B(0, R')$

$$\int_\Omega U_i^5 \leq \frac{4\pi}{\lambda_i^{1/2}} \int_0^{2R'\lambda_i} \frac{r^2}{(1+r^2)^{5/2}} dr = O(\frac{1}{\lambda_i^{1/2}}) \tag{C.13}$$

and

$$\int_\Omega \frac{\mu e^{-\mu^{1/2}|x-a_i|}}{\lambda^{1/2}|x-a_i|} dx \leq \frac{4\pi}{\lambda_i^{1/2}} \int_0^{2R'} \mu r e^{-\mu^{1/2} r} dr = O(\frac{1}{\lambda_i^{1/2}}).$$

Therefore

$$\int_{\Omega\setminus B_j} \left(3U_i^5 - \frac{\mu e^{-\mu^{1/2}|x-a_i|}}{\lambda^{1/2}|x-a_i|}\right) V_j = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}). \tag{C.14}$$

Finally, we notice that

$$U_i^5 = O(\frac{\mu^{5/2}}{\lambda_i^{5/2}}) \qquad \frac{\mu e^{-\mu^{1/2}|x-a_i|}}{\lambda^{1/2}|x-a_i|} = O(\frac{\mu^{3/2}}{\lambda_i^{1/2}}) \qquad \text{in } B_j$$

and

$$\begin{aligned} \int_{B_j} |V_j| &\leq \frac{4\pi}{\lambda_j^{5/2}} \int_0^{\frac{\gamma\lambda_j}{4\mu^{1/2}}} \left| \frac{1}{(1+r^2)^{1/2}} - \frac{1-e^{-(\mu^{1/2}/\lambda_j)r}}{r} \right| r^2 dr \\ &\leq \frac{4\pi}{\lambda_j^{5/2}} \left(\int_0^{\frac{\gamma\lambda_j}{4\mu^{1/2}}} (r - \frac{r^2}{(1+r^2)^{1/2}}) dr + \int_0^{\frac{\gamma\lambda_j}{4\mu^{1/2}}} r e^{-(\mu^{1/2}/\lambda_j)r} dr \right). \end{aligned}$$

Consequently

$$\int_{B_j} |V_j| = O\big(\frac{\ln(\lambda_j/\mu^{1/2})}{\lambda_j^{5/2}} + \frac{1}{\lambda_j^{1/2}\mu}\big) \tag{C.15}$$

and

$$\int_{B_j} \left(3U_i^5 - \frac{\mu e^{-\mu^{1/2}|x-a_i|}}{\lambda^{1/2}|x-a_i|}\right) V_j = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}).$$

This estimate, joined to (C.10-11,16), yield (C.9).

The other quantities in (3.13), with $i \neq j$, may be estimated in the same way. Similar computations also yield

$$\left(\int_\Omega |V_i|^{24/5}|V_j|^{6/5}\right)^{5/6} = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}). \tag{C.16}$$

Indeed, we have

$$\int_\Omega |V_i|^{24/5} \leq \frac{4\pi}{\lambda_i^{3/5}}\int_0^{2R'\lambda_i}\left|\frac{1}{(1+r^2)^{1/2}} - \frac{1-e^{-(\mu^{1/2}/\lambda_i)r}}{r}\right|^{24/5} r^2 dr$$

and the integral on the right hand side goes to a finite limit as $\mu^{1/2}/\lambda_i$ goes to zero, whence

$$\left(\int_\Omega |V_i|^{24/5}\right)^{5/6} = O(\frac{1}{\lambda_i^{1/2}})$$

and, taking account of (C.12)

$$\left(\int_{\Omega\setminus B_j} |V_i|^{24/5}|V_j|^{6/5}\right)^{5/6} = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}). \tag{C.17}$$

On the other hand

$$\begin{aligned}\int_{B_j} |V_j|^{6/5} &\leq \frac{4\pi}{\lambda_j^{5/2}}\int_0^{\frac{\gamma\lambda_j}{4\mu^{1/2}}}\left|\frac{1}{(1+r^2)^{1/2}} - \frac{1-e^{-(\mu^{1/2}/\lambda_j)r}}{r}\right|^{6/5} r^2 dr \\ &= O\left(\frac{1}{\lambda_j^{12/5}}\int_0^{\frac{\gamma\lambda_j}{4\mu^{1/2}}}\left(\frac{1}{r^3} + \frac{e^{-(\mu^{1/2}/\lambda_j)r}}{r}\right)^{6/5} r^2 dr\right)\end{aligned}$$

and

$$\int_0^{\frac{\gamma\lambda_j}{4\mu^{1/2}}} r^{4/5}e^{-(6\mu^{1/2}/5\lambda_j)r}dr = (\frac{\lambda_j}{\mu^{1/2}})^{9/5}\int_0^{\gamma/4} t^{4/5}e^{-6t/5}dt = O(\frac{\lambda_j^{9/5}}{\mu^{9/10}}).$$

Therefore

$$\left(\int_\Omega |V_j|^{6/5}\right)^{5/6} = O(\frac{1}{\lambda_j^{1/2}\mu^{3/4}})$$

and (C.12) applied to V_i in B_j provides us with

$$\left(\int_{B_j} |V_i|^{24/5}|V_j|^{6/5}\right)^{5/6} = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}\cdot\frac{\mu^{3/4}}{\lambda_i^{3/2}}). \tag{C.18}$$

(C.17-18) prove (C.16).

We state now :

Proposition C.1 *Let $(\alpha,\lambda,a)\in\tilde{\mathcal{N}}_\mu$ (i.e. $|\alpha_i-3^{1/4}|<\eta_1$, $\mu^{1/2}/\lambda_i<\eta_1$, $d(a_i,\partial\Omega)>\rho$, $|a_i-a_j|>\frac{\gamma}{2\mu^{1/2}}$ if $i\neq j$). We have*

$$\frac{\partial K_\mu}{\partial\alpha_i}(\alpha,\lambda,a,\bar{v}_\mu(\alpha,\lambda,a))=\frac{3\pi^2}{4}\alpha_i(3-\alpha_i^4)+O\left(\frac{\mu^{1/2}}{|\lambda|}\right) \tag{C.19}$$

$$\begin{aligned}\frac{\partial K_\mu}{\partial\lambda_i}(\alpha,\lambda,a,\bar{v}_\mu(\alpha,\lambda,a))&=\pi(2-\frac{4}{3}\alpha^4)\frac{\mu^{1/2}}{\lambda_i^2}\\&+O\left(\frac{\mu^{1/2}}{\lambda_i^{3/2}|\lambda|^{1/2}}\sum_l|3-\alpha_l^4|\right)+o\left(\frac{\mu^{1/2}}{\lambda_i^{3/2}|\lambda|^{1/2}}\right)\end{aligned} \tag{C.20}$$

$$\frac{\partial K_\mu}{\partial(a_i)_j}(\alpha,\lambda,a,\bar{v}_\mu(\alpha,\lambda,a))=O\left(\frac{\mu^{1/2}\lambda_i}{|\lambda|}\right) \tag{C.21}$$

$$\frac{\partial^2 K_\mu}{\partial\alpha_i^2}(\alpha,\lambda,a,\bar{v}_\mu(\alpha,\lambda,a))=\frac{3\pi^2}{4}(3-5\alpha_i^4)+O\left(\frac{\mu^{1/2}}{|\lambda|}\right) \tag{C.22}$$

as $\mu\to\infty$ and $\mu^{1/2}/\lambda_i\to\infty$, $1\le i\le k$.

Let us prove (C.19). According to (3.3-4), we have for $(\alpha,\lambda,a,v)\in\mathcal{N}_\mu$

$$\begin{aligned}\frac{\partial K_\mu}{\partial\alpha_i}(\alpha,\lambda,a,v)=&\int_\Omega\nabla(\sum_j\alpha_jV_j).\nabla V_i+\mu\int_\Omega(\sum_j\alpha_jV_j+v)V_i\\&-\int_\Omega(\sum_j\alpha_jV_j+v)^5V_i.\end{aligned}$$

Let us expand $(\sum_j\alpha_jV_j+v)^5$. (3.15-16) and (C.16) provide us with

$$\begin{aligned}&\left|\mu\int_\Omega V_iv-\int_\Omega(\sum_j\alpha_jV_j+v)^4V_iv\right|\\&\qquad\le C\frac{\mu^{1/2}}{\lambda_i^{1/2}}\left(\frac{1}{|\lambda|^{1/2}}+\frac{1}{\mu^{3/4}}\right)\left(\int_\Omega(|\nabla v|^2+\mu v^2)\right)^{1/2}.\end{aligned}$$

Moreover, Hölder inequality, Sobolev embedding theorem and (C.3) show that the contribution to the integral of the terms in which v occurs with an exponent larger or equal to 2 is dominated by $\|v\|^2_{H^1(\Omega)}$. As a consequence, if $(\alpha,\lambda,a)\in\tilde{\mathcal{N}}_\mu$ and $v=\bar{v}_\mu(\alpha,\lambda,a)$, (3.12) yields

$$\begin{aligned}\frac{\partial K_\mu}{\partial\alpha_i}(\alpha,\lambda,a,\bar{v}_\mu)=&\frac{\partial K_\mu}{\partial\alpha_i}(\alpha,\lambda,a,0)\\&+O\left(\frac{\mu^{1/2}}{\lambda_i^{1/2}|\lambda|^{3/2}}+\frac{1}{\mu^{1/2}\lambda_i^{1/2}|\lambda|^{1/2}}\right).\end{aligned} \tag{C.23}$$

In view of (C.1-3), we have

$$\alpha_i \int_\Omega |\nabla V_i|^2 + \mu\alpha_i \int_\Omega V_i^2 - \alpha_i^5 \int_\Omega V_i^6 = \frac{\pi^2}{4}\alpha_i(3-\alpha_i^4) + O(\frac{\mu^{1/2}}{\lambda_i}).$$

We recall that, according to (C.9), we have also

$$\int_\Omega \nabla V_i . \nabla V_j = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}).$$

Similar computations show that

$$\int_\Omega V_i^5 V_j = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}})$$

and

$$\int_\Omega \left((\sum_j \alpha_j V_j)^5 V_i - \sum_j \alpha_j^5 V_j^5 V_i \right) = o(\frac{\mu^{1/2}}{\lambda_i^{1/2}|\lambda|^{1/2}}).$$

Lastly, from (C.12,15) we deduce that

$$\mu \int_\Omega V_i V_j = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}).$$

Collecting thse results, (C.19) is established.

(C.21-22) follow from the same kind of integral estimates. The only result which requires to be more careful is (C.20). From (3.3-4), we have

$$\begin{aligned}\frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial \lambda_i}(\alpha,\lambda,a,v) = &\int_\Omega \nabla(\sum_j \alpha_j V_j).\nabla \frac{\partial V_i}{\partial \lambda_i} \\ &+ \mu \int_\Omega (\sum_j \alpha_j V_j + v)\frac{\partial V_i}{\partial \lambda_i} - \int_\Omega (\sum_j \alpha_j V_j + v)^5 \frac{\partial V_i}{\partial \lambda_i}\end{aligned}$$

for $(\alpha,\lambda,a,v) \in \mathcal{N}_\mu$. The same arguments as in the proof of Lemma 3.1 in [40] show that

$$\begin{aligned}&\left| \mu \int_\Omega \frac{\partial V_i}{\partial \lambda_i} v - \alpha_i^4 \int_\Omega V_i^4 \frac{\partial V_i}{\partial \lambda_i} v \right| \\ &\qquad \leq C \frac{\mu^{1/2}}{\lambda_i^{3/2}} \left(\frac{1}{|\lambda|^{1/2}} + \frac{1}{\mu^{3/4}} \right) \left(\int_\Omega (|\nabla v|^2 + \mu v^2) \right)^{1/2}.\end{aligned}$$

We have also, if $i \neq j$

$$\left| \int_\Omega V_i^4 \frac{\partial V_i}{\partial \lambda_i} v \right| \leq C \|v\|_{H^1(\Omega)} \left(\int_\Omega |V_j|^{24/5} \left| \frac{\partial V_i}{\partial \lambda_i} \right|^{6/5} \right)^{5/6}$$

and proceeding as in the proof of (C.16), we find

$$\left(\int_\Omega |V_j|^{24/5}\left|\frac{\partial V_i}{\partial \lambda_i}\right|^{6/5}\right)^{5/6} = O(\frac{\mu^{1/2}}{\lambda_i^{3/2}\lambda_j^{1/2}}).$$

Lastly, using Hölder inequality, Sobolev embedding theorem and integral estimates as the previous one, we see that the terms in which v occurs with an exponent larger or equal to 2 have a contribution to the integral which is dominated by $\|v\|^2_{H^1(\Omega)}/\lambda_i$. Therefore, similarly to (C.23) we obtain, using (3.12)

$$\frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial \alpha_i}(\alpha,\lambda,a,\bar{v}_\mu) = \frac{1}{\alpha_i}\frac{\partial K_\mu}{\partial \alpha_i}(\alpha,\lambda,a,0) + O\left(\frac{\mu^{1/2}}{\lambda_i^{3/2}|\lambda|^{3/2}} + \frac{1}{\mu^{1/2}\lambda_i^{3/2}|\lambda|^{1/2}}\right)$$

for any $(\alpha,\lambda,a)\in\tilde{\mathcal{N}}_\mu$. We notice that, according to (C.4-6)

$$\int_\Omega \nabla V_i.\nabla\frac{\partial V_i}{\partial\lambda_i} + \mu\int_\Omega V_i\frac{\partial V_i}{\partial \lambda_i} - \alpha_i^4\int_\Omega V_i^5\frac{\partial V_i}{\partial\lambda_i}$$
$$= \pi(2-\frac{4}{3}\alpha^4)\frac{\mu^{1/2}}{\lambda_i^2} + O\left(\frac{1}{\lambda_i^2\mu^{1/2}}\sum_l |3-\alpha_l^4|\right).$$

In addition, using (2.15), we have

$$\int_\Omega \nabla(\alpha_j V_j).\nabla\frac{\partial V_i}{\partial\lambda_i} + \mu\int_\Omega \alpha_j V_j\frac{\partial V_i}{\partial\lambda_i} - \int_\Omega (\alpha_j V_j)^5\frac{\partial V_i}{\partial \lambda_i}$$
$$= \alpha_j\left(\int_{\partial\Omega}\frac{\partial V_j}{\partial\nu}\frac{\partial V_i}{\partial\lambda_i} + \int_\Omega\left(3U_j^5 - \alpha_j^4 V_j^5 + \mu(U_j - \frac{1}{\lambda_j^{1/2}|x-a_j|})\right)\frac{\partial V_i}{\partial\lambda_i}\right).$$

As a_i and a_j do not approach the boundary of Ω, the definition of V_i, V_j shows that

$$\frac{\partial V_j}{\partial\nu} = o(\frac{\mu^{1/2}}{\lambda_j^{1/2}}) \qquad \frac{\partial V_i}{\partial\lambda_i} = o(\frac{1}{\lambda_i^{3/2}}) \qquad \text{uniformly on } \partial\Omega$$

whence

$$\int_{\partial\Omega}\frac{\partial V_j}{\partial\nu}\frac{\partial V_i}{\partial\lambda_i} = O(\frac{\mu^{1/2}}{\lambda_i^{3/2}\lambda_j^{1/2}}).$$

On the other hand, similarly to (C.2), we have

$$\int_\Omega\left(\frac{\partial V_i}{\partial\lambda_i}\right)^2 = O(\frac{1}{\lambda_i^3\mu^{1/2}})$$

and

$$\int_\Omega\left(U_j - \frac{1}{\lambda_j^{1/2}|x-a_j|}\right)^2 \le \frac{4\pi}{\lambda_j^2}\int_0^{2R'\lambda_j}\left(\frac{1}{(1+r^2)^{1/2}} - \frac{1}{r}\right)^2 r^2 dr = O(\frac{1}{\lambda_j^2})$$

whence, using Schwarz inequality

$$\mu \int_\Omega \left(U_j - \frac{1}{\lambda_j^{1/2}|x-a_j|} \right) \frac{\partial V_i}{\partial \lambda_i} = O(\frac{\mu^{3/4}}{\lambda_i^{3/2}\lambda_j}).$$

Lastly, we write

$$\int_\Omega (3U_j^5 - \alpha_j^4 V_j^5) \frac{\partial V_i}{\partial \lambda_i} = 3 \int_\Omega (U_j^5 - V_j^5) \frac{\partial V_i}{\partial \lambda_i} + (3-\alpha_j^4) \int_\Omega V_j^5 \frac{\partial V_i}{\partial \lambda_i}.$$

Proceeding as in the proof of (C.10), we find

$$\int_\Omega V_j^5 \frac{\partial V_i}{\partial \lambda_i} = O(\frac{\mu^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}}).$$

Concerning the last integral that we have to estimate, setting as previously $B_l = B(a_l, \gamma/4\mu^{1/2})$, we have, similarly to (C.15)

$$\int_{B_i} \left| \frac{\partial V_i}{\partial \lambda_i} \right| = O(\frac{\ln(\lambda_i/\mu^{1/2})}{\lambda_i^{7/2}} + \frac{1}{\lambda_i^{3/2}\mu})$$

and

$$\begin{aligned} &\int_\Omega |U_j^5 - V_j^5| \\ &= O\left(\int_\Omega (U_j^4 \varphi_j + \varphi_j^5) \right) \\ &= O\left(\frac{1}{\lambda_j^{1/2}} \int_0^{2R'\lambda_j} \frac{1-e^{-(\mu^{1/2}/\lambda_j)r}}{(1+r^2)^2} r^2 dr + \frac{\mu}{\lambda_j^{5/2}} \int_0^{2R'\mu^{1/2}} \frac{(1-e^{-r})^5}{r^3} dr \right) \\ &= O(\frac{\mu^{1/2}}{\lambda_j^{3/2}}). \end{aligned}$$

As we have also

$$\frac{\partial V_i}{\partial \lambda_i} = O(\frac{\mu^{1/2}}{\lambda_j^{3/2}}) \text{ outside of } B_i \qquad U_j^5 - V_j^5 = O(\frac{\mu^{5/2}}{\lambda_j^{5/2}}) \text{ in } B_i$$

we obtain

$$\int_\Omega (U_j^5 - V_j^5) \frac{\partial V_i}{\partial \lambda_i} = O(\frac{\mu}{\lambda_i^{3/2}\lambda_j^{3/2}})$$

and (C.20) is proved.

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